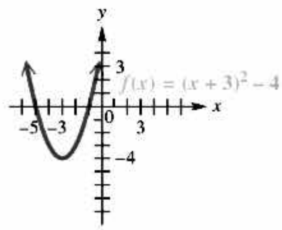
*Consider the graph of each quadratic function. Do the following.* ***See Examples 1–4.***

***(a)*** *Give the domain and range.* ***(b)*** *Give the coordinates of the vertex.*

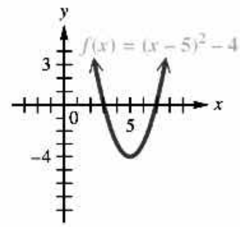
***(c)*** *Give the equation of the axis.* ***(d)*** *Find the y-intercept.*

***(e)*** *Find the x-intercepts.*

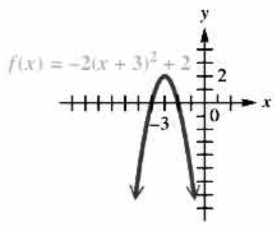
**11.**



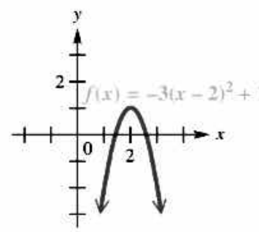

**12.**




**13.**




**14.**




***Concept Check*** *Match each function with its graph without actually entering it into a calculator. Then, after completing the exercises, check the answers with a calculator. Use the standard viewing window.*

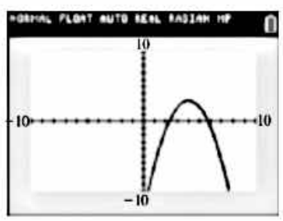
**15.** $f(x) = (x - 4)^2 - 3$

**16.** $f(x) = -(x - 4)^2 + 3$

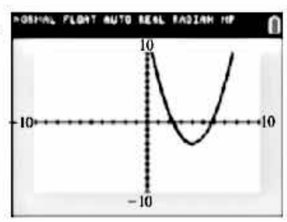
**17.** $f(x) = (x + 4)^2 - 3$

**18.** $f(x) = -(x + 4)^2 + 3$

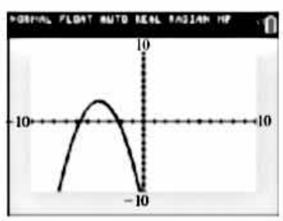
**A.**



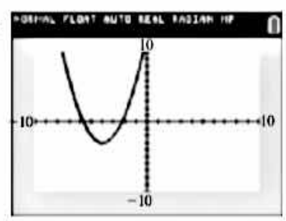

**B.**




**C.**




**D.**




**19.** Graph the following on the same coordinate system.

**(a)** $y = x^2$ **(b)** $y = 3x^2$ **(c)** $y = \frac{1}{3}x^2$

**(d)** How does the coefficient of $x^2$ affect the shape of the graph?

**20.** Graph the following on the same coordinate system.

**(a)** $y = x^2$ **(b)** $y = x^2 - 2$ **(c)** $y = x^2 + 2$

**(d)** How do the graphs in parts (b) and (c) differ from the graph of $y = x^2$?

**21.** Graph the following on the same coordinate system.

**(a)** $y = (x - 2)^2$ **(b)** $y = (x + 1)^2$ **(c)** $y = (x + 3)^2$

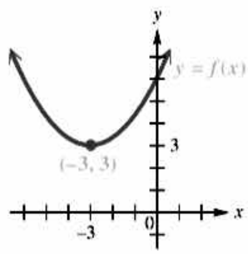
**(d)** How do these graphs differ from the graph of $y = x^2$?

**22.** *Concept Check* A quadratic function $f(x)$ has vertex $(0, 0)$, and all of its intercepts are the same point. What is the general form of its equation?

*Graph each quadratic function. Give the **(a)** vertex, **(b)** axis, **(c)** domain, and **(d)** range. Then determine **(e)** the largest open interval of the domain over which the function is increasing and **(f)** the largest open interval over which the function is decreasing.* ***See Examples 1–4.***

| | |
|---|---|
| **23.** $f(x) = (x - 2)^2$ | **24.** $f(x) = (x + 4)^2$ |
| **25.** $f(x) = (x + 3)^2 - 4$ | **26.** $f(x) = (x - 5)^2 - 4$ |
| **27.** $f(x) = -\frac{1}{2}(x + 1)^2 - 3$ | **28.** $f(x) = -3(x - 2)^2 + 1$ |
| **29.** $f(x) = x^2 - 2x + 3$ | **30.** $f(x) = x^2 + 6x + 5$ |
| **31.** $f(x) = x^2 - 10x + 21$ | **32.** $f(x) = 2x^2 - 4x + 5$ |
| **33.** $f(x) = -2x^2 - 12x - 16$ | **34.** $f(x) = -3x^2 + 24x - 46$ |
| **35.** $f(x) = -\frac{1}{2}x^2 - 3x - \frac{1}{2}$ | **36.** $f(x) = \frac{2}{3}x^2 - \frac{8}{3}x + \frac{5}{3}$ |

*Concept Check The figure shows the graph of a quadratic function $y = f(x)$. Use it to answer each question.*




**37.** What is the minimum value of $f(x)$?

**38.** For what value of $x$ is $f(x)$ as small as possible?

**39.** How many real solutions are there to the equation $f(x) = 1$?

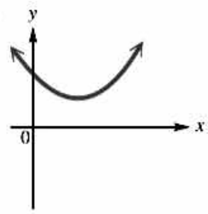
**40.** How many real solutions are there to the equation $f(x) = 4$?

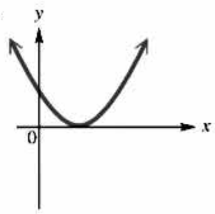
*Concept Check Several graphs of the quadratic function*

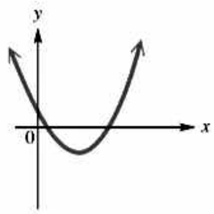
$$f(x) = ax^2 + bx + c$$

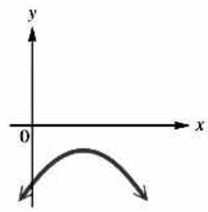
*are shown below. For the given restrictions on a, b, and c, select the corresponding graph from choices A–F. (Hint: Use the discriminant.)*

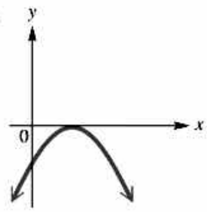
| | |
|---|---|
| **41.** $a < 0$; $b^2 - 4ac = 0$ | **42.** $a > 0$; $b^2 - 4ac < 0$ |
| **43.** $a < 0$; $b^2 - 4ac < 0$ | **44.** $a < 0$; $b^2 - 4ac > 0$ |
| **45.** $a > 0$; $b^2 - 4ac > 0$ | **46.** $a > 0$; $b^2 - 4ac = 0$ |

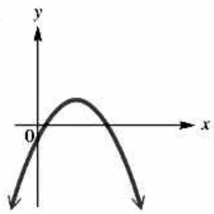
**A.**


**B.**


**C.**


**D.**


**E.**


**F.**


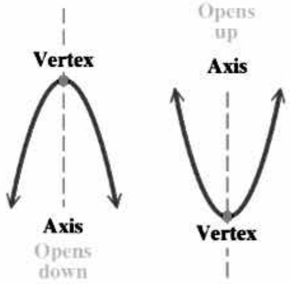



**Figure 2**

Parabolas are symmetric with respect to a line (the $y$-axis in **Figure 1**). This line is the **axis of symmetry,** or **axis,** of the parabola. The point where the axis intersects the parabola is the **vertex** of the parabola. As **Figure 2** shows, the vertex of a parabola that opens down is the highest point of the graph, and the vertex of a parabola that opens up is the lowest point of the graph.

**Graphing Techniques** Graphing techniques may be applied to the graph of $f(x) = x^2$ to give the graph of a different quadratic function. Compared to the basic graph of $f(x) = x^2$, the graph of $F(x) = a(x - h)^2 + k$, with $a \neq 0$, has the following characteristics.

$$F(x) = a(x - h)^2 + k$$

| | Horizontal shift: | Vertical shift: |
|---|---|---|
| • Opens up if $a > 0$ | • $h$ units right if $h > 0$ | • $k$ units up if $k > 0$ |
| • Opens down if $a < 0$ | • $\lvert h \rvert$ units left if $h < 0$ | • $\lvert k \rvert$ units down if $k < 0$ |
| • Vertically stretched (narrower) if $\lvert a \rvert > 1$ | | |
| • Vertically shrunk (wider) if $0 < \lvert a \rvert < 1$ | | |

### EXAMPLE 1 Graphing Quadratic Functions

Graph each function. Give the domain and range.

**(a)** $f(x) = x^2 - 4x - 2$ (by plotting points)

**(b)** $g(x) = -\frac{1}{2}x^2$ (and compare to $y = x^2$ and $y = \frac{1}{2}x^2$)

**(c)** $F(x) = -\frac{1}{2}(x - 4)^2 + 3$ (and compare to the graph in part (b))

**SOLUTION**

**(a)** See the table with **Figure 3.** The domain of $f(x) = x^2 - 4x - 2$ is $(-\infty, \infty)$, the range is $[-6, \infty)$, the vertex is the point $(2, -6)$, and the axis has equation $x = 2$. **Figure 4** shows how a graphing calculator displays this graph.

| $x$ | $f(x)$ |
|---|---|
| $-1$ | 3 |
| 0 | $-2$ |
| 1 | $-5$ |
| 2 | $-6$ |
| 3 | $-5$ |
| 4 | $-2$ |
| 5 | 3 |

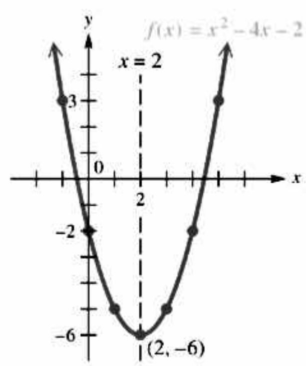



**Figure 3**

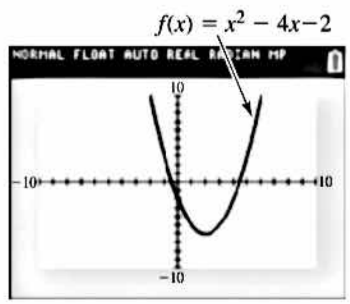



**Figure 4**

**(b)** Think of $g(x) = -\frac{1}{2}x^2$ as $g(x) = -\left(\frac{1}{2}x^2\right)$. The graph of $y = \frac{1}{2}x^2$ is a wider version of the graph of $y = x^2$, and the graph of $g(x) = -\left(\frac{1}{2}x^2\right)$ is a reflection of the graph of $y = \frac{1}{2}x^2$ across the $x$-axis. See **Figure 5** on the next page. The vertex is the point $(0, 0)$, and the axis of the parabola is the line $x = 0$ (the $y$-axis). The domain is $(-\infty, \infty)$, and the range is $(-\infty, 0]$.

Calculator graphs are shown in **Figure 6.**

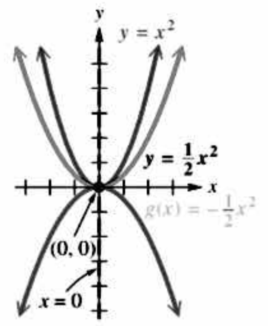



**Figure 5**

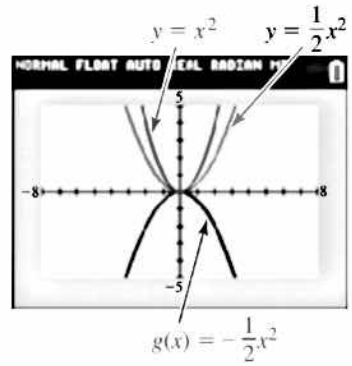



**Figure 6**

**(c)** Notice that $F(x) = -\frac{1}{2}(x - 4)^2 + 3$ is related to $g(x) = -\frac{1}{2}x^2$ from part (b). The graph of $F(x)$ is the graph of $g(x)$ translated 4 units to the right and 3 units up. See **Figure 7.** The vertex is the point $(4, 3)$, which is also shown in the calculator graph in **Figure 8,** and the axis of the parabola is the line $x = 4$. The domain is $(-\infty, \infty)$, and the range is $(-\infty, 3]$.

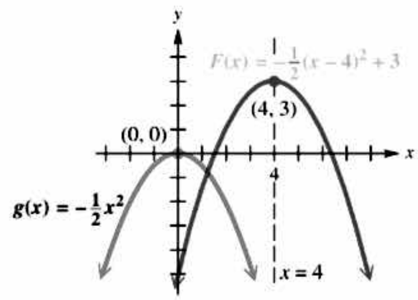



**Figure 7**

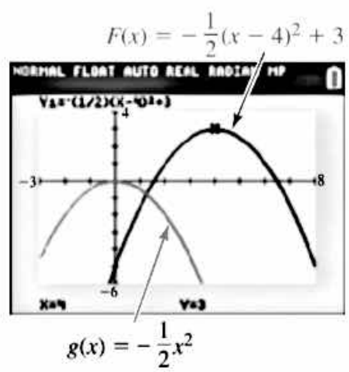



**Figure 8**

**✔ Now Try Exercises 19 and 21.**

## Completing the Square

In general, the graph of the quadratic function

$$f(x) = a(x - h)^2 + k \quad (a \neq 0)$$

is a parabola with ***vertex*** $(h, k)$ and ***axis of symmetry*** $x = h$. The parabola opens up if $a$ is positive and down if $a$ is negative. With these facts in mind, we *complete the square* to graph the general quadratic function

$$f(x) = ax^2 + bx + c.$$

### EXAMPLE 2 Graphing a Parabola ($a = 1$)

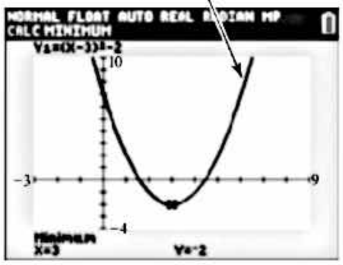
Graph $f(x) = x^2 - 6x + 7$. Find the largest open intervals over which the function is increasing or decreasing.

**SOLUTION** We express $x^2 - 6x + 7$ in the form $(x - h)^2 + k$ by completing the square. In preparation for this, we first write

$$f(x) = (x^2 - 6x \qquad\qquad) + 7. \quad \text{Prepare to complete the square.}$$

We must add a number inside the parentheses to obtain a perfect square trinomial. Find this number by taking half the coefficient of $x$ and squaring the result.

$f(x) = x^2 - 6x + 7$
$f(x) = (x - 3)^2 - 2$




This screen shows that the vertex of the graph in **Figure 9** is the point $(3, -2)$. Because it is the *lowest* point on the graph, we direct the calculator to find the *minimum*.

$$\left[\tfrac{1}{2}(-6)\right]^2 = (-3)^2 = 9 \quad \text{Take half the coefficient of } x. \text{ Square the result.}$$

$$f(x) = (x^2 - 6x + 9 - 9) + 7 \quad \text{Add and subtract 9.}$$

This is the same as adding 0.

$$f(x) = (x^2 - 6x + 9) - 9 + 7 \quad \text{Regroup terms.}$$

$$f(x) = (x - 3)^2 - 2 \quad \text{Factor and simplify.}$$

The vertex of the parabola is the point $(3, -2)$, and the axis is the line $x = 3$. We find additional ordered pairs that satisfy the equation, as shown in the table, and plot and join these points to obtain the graph in **Figure 9.**

| $x$ | $y$ | |
|---|---|---|
| 0 | 7 | ← $y$-intercept |
| 1 | 2 | |
| 3 | −2 | ← Vertex |
| 5 | 2 | Find using symmetry |
| 6 | 7 | about the axis. |

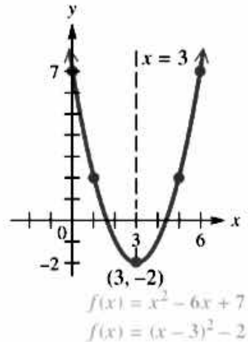



**Figure 9**

The domain of this function is $(-\infty, \infty)$, and the range is $[-2, \infty)$. Because the lowest point on the graph is the vertex $(3, -2)$, the function is decreasing on $(-\infty, 3)$ and increasing on $(3, \infty)$.

✔ **Now Try Exercise 31.**

NOTE In **Example 2** we added and subtracted 9 *on the same side* of the equation to complete the square. This differs from adding the same number to *each side of the equation*, as is sometimes done in the procedure. We want $f(x)$—that is, $y$—alone on one side of the equation, so we adjusted that step in the process of completing the square here.

## EXAMPLE 3 Graphing a Parabola ($a \neq 1$)

Graph $f(x) = -3x^2 - 2x + 1$. Identify the intercepts of the graph.

**SOLUTION** To complete the square, the coefficient of $x^2$ must be 1.

$$f(x) = -3\left(x^2 + \frac{2}{3}x \qquad \right) + 1 \quad \text{Factor } -3 \text{ from the first two terms.}$$

$$f(x) = -3\left(x^2 + \frac{2}{3}x + \frac{1}{9} - \frac{1}{9}\right) + 1 \quad \left[\tfrac{1}{2}\left(\tfrac{2}{3}\right)\right]^2 = \left(\tfrac{1}{3}\right)^2 = \tfrac{1}{9}, \text{ so add and subtract } \tfrac{1}{9}.$$

$$f(x) = -3\left(x^2 + \frac{2}{3}x + \frac{1}{9}\right) - 3\left(-\frac{1}{9}\right) + 1 \quad \text{Distributive property}$$

Be careful here.

$$f(x) = -3\left(x + \frac{1}{3}\right)^2 + \frac{4}{3} \quad \text{Factor and simplify.}$$

The vertex is the point $\left(-\frac{1}{3}, \frac{4}{3}\right)$. The intercepts are good additional points to find. The $y$-intercept is found by evaluating $f(0)$.

$$f(0) = -3(0)^2 - 2(0) + 1 \quad \text{Let } x = 0 \text{ in } f(x) = -3x^2 - 2x + 1.$$

$$f(0) = 1 \longleftarrow \text{The } y\text{-intercept is } (0, 1).$$

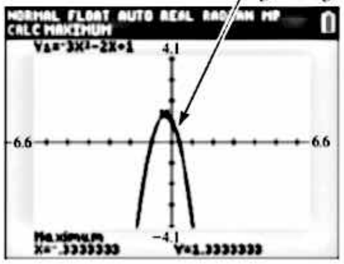
The $x$-intercepts are found by setting $f(x)$ equal to 0 and solving for $x$.

$$0 = -3x^2 - 2x + 1 \quad \text{Set } f(x) = 0.$$
$$0 = 3x^2 + 2x - 1 \quad \text{Multiply by } -1.$$
$$0 = (3x - 1)(x + 1) \quad \text{Factor.}$$
$$x = \frac{1}{3} \quad \text{or} \quad x = -1 \quad \text{Zero-factor property}$$

Therefore, the $x$-intercepts are $\left(\frac{1}{3}, 0\right)$ and $(-1, 0)$. The graph is shown in **Figure 10.**

$f(x) = -3x^2 - 2x + 1$

$f(x) = -3\left(x + \frac{1}{3}\right)^2 + \frac{4}{3}$




This screen gives the vertex of the graph in **Figure 10** as the point $\left(-\frac{1}{3}, \frac{4}{3}\right)$. (The display shows decimal approximations.) We want the highest point on the graph, so we direct the calculator to find the *maximum*.

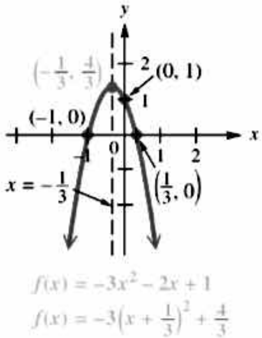



**Figure 10**

✔ **Now Try Exercise 33.**

---

**NOTE** It is possible to reverse the process of **Example 3** and write the quadratic function from its graph in **Figure 10** if the vertex and any other point on the graph are known. Because quadratic functions take the form

$$f(x) = a(x - h)^2 + k,$$

we can substitute the $x$- and $y$-values of the vertex, $\left(-\frac{1}{3}, \frac{4}{3}\right)$, for $h$ and $k$.

$$f(x) = a\left[x - \left(-\frac{1}{3}\right)\right]^2 + \frac{4}{3} \quad \text{Let } h = -\tfrac{1}{3} \text{ and } k = \tfrac{4}{3}.$$
$$f(x) = a\left(x + \frac{1}{3}\right)^2 + \frac{4}{3} \quad \text{Simplify.}$$

We find the value of $a$ by substituting the $x$- and $y$-coordinates of any other point on the graph, say $(0, 1)$, into this function and solving for $a$.

$$1 = a\left(0 + \frac{1}{3}\right)^2 + \frac{4}{3} \quad \text{Let } x = 0 \text{ and } y = 1.$$
$$1 = a\left(\frac{1}{9}\right) + \frac{4}{3} \quad \text{Square.}$$
$$-\frac{1}{3} = \frac{1}{9}a \quad \text{Subtract } \tfrac{4}{3}.$$
$$a = -3 \quad \text{Multiply by 9. Interchange sides.}$$

Verify in **Example 3** that the vertex form of the quadratic function is

$$f(x) = -3\left(x + \frac{1}{3}\right)^2 + \frac{4}{3}.$$

Exercises of this type are labeled ***Connecting Graphs with Equations.***

---

**LOOKING AHEAD TO CALCULUS**

An important concept in calculus is the **definite integral.** If the graph of $f$ lies above the $x$-axis, the symbol

$$\int_a^b f(x)\,dx$$

represents the area of the region above the $x$-axis and below the graph of $f$ from $x = a$ to $x = b$. For example, in **Figure 10** with

$$f(x) = -3x^2 - 2x + 1,$$

$a = -1$, and $b = \frac{1}{3}$, calculus provides the tools for determining that the area enclosed by the parabola and the $x$-axis is $\frac{32}{27}$ (square units).

**The Vertex Formula** We can generalize the earlier work to obtain a formula for the vertex of a parabola.

$$f(x) = ax^2 + bx + c \qquad \text{General quadratic form}$$

$$= a\left(x^2 + \frac{b}{a}x \qquad\right) + c \qquad \text{Factor } a \text{ from the first two terms.}$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - a\left(\frac{b^2}{4a^2}\right) \qquad \text{Add } \left[\frac{1}{2}\left(\frac{b}{a}\right)\right]^2 = \frac{b^2}{4a^2} \text{ inside the parentheses. Subtract } a\left(\frac{b^2}{4a^2}\right) \text{ outside the parentheses.}$$

$$= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \qquad \text{Factor and simplify.}$$

$$f(x) = a\left[x - \underbrace{\left(-\frac{b}{2a}\right)}_{h}\right]^2 + \underbrace{\frac{4ac - b^2}{4a}}_{k} \qquad \text{Vertex form of } f(x) = a(x - h)^2 + k$$

Thus, the vertex $(h, k)$ can be expressed in terms of $a$, $b$, and $c$. ***It is not necessary to memorize the expression for k because it is equal to $f(h) = f\left(-\frac{b}{2a}\right)$.***

## Graph of a Quadratic Function

The quadratic function $f(x) = ax^2 + bx + c$ can be written as

$$y = f(x) = a(x - h)^2 + k, \quad \text{with } a \neq 0,$$

where $$h = -\frac{b}{2a} \quad \text{and} \quad k = f(h). \qquad \text{Vertex formula}$$

The graph of $f$ has the following characteristics.

1. It is a parabola with vertex $(h, k)$ and the vertical line $x = h$ as axis.
2. It opens up if $a > 0$ and down if $a < 0$.
3. It is wider than the graph of $y = x^2$ if $|a| < 1$ and narrower if $|a| > 1$.
4. The $y$-intercept is $(0, f(0)) = (0, c)$.
5. The $x$-intercepts are found by solving the equation $ax^2 + bx + c = 0$.
   - If $b^2 - 4ac > 0$, then the $x$-intercepts are $\left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, 0\right)$.
   - If $b^2 - 4ac = 0$, then the $x$-intercept is $\left(-\frac{b}{2a}, 0\right)$.
   - If $b^2 - 4ac < 0$, then there are no $x$-intercepts.

### EXAMPLE 4 Using the Vertex Formula

Find the axis and vertex of the parabola having equation $f(x) = 2x^2 + 4x + 5$.

**SOLUTION** The axis of the parabola is the vertical line

$$x = h = -\frac{b}{2a} = -\frac{4}{2(2)} = -1. \qquad \text{Use the vertex formula. Here } a = 2 \text{ and } b = 4.$$

The vertex is $(-1, f(-1))$. Evaluate $f(-1)$.

$$f(-1) = 2(-1)^2 + 4(-1) + 5 = 3$$

The vertex is $(-1, 3)$. **✔ Now Try Exercise 31(a).**

**Quadratic Models** Because the vertex of a vertical parabola is the highest or lowest point on the graph, equations of the form

$$y = ax^2 + bx + c$$

are important in certain problems where we must find the maximum or minimum value of some quantity.

- When $a < 0$, the $y$-coordinate of the vertex gives the maximum value of $y$.
- When $a > 0$, the $y$-coordinate of the vertex gives the minimum value of $y$.

The $x$-coordinate of the vertex tells *where* the maximum or minimum value occurs.

If air resistance is neglected, the height $s$ (in feet) of an object projected directly upward from an initial height $s_0$ feet with initial velocity $v_0$ feet per second is

$$s(t) = -16t^2 + v_0t + s_0,$$

where $t$ is the number of seconds after the object is projected. The coefficient of $t^2$ (that is, $-16$) is a constant based on the gravitational force of Earth. This constant is different on other surfaces, such as the moon and the other planets.

## EXAMPLE 5 Solving a Problem Involving Projectile Motion

A ball is projected directly upward from an initial height of 100 ft with an initial velocity of 80 ft per sec.

**(a)** Give the function that describes the height of the ball in terms of time $t$.

**(b)** After how many seconds does the ball reach its maximum height? What is this maximum height?

**(c)** For what interval of time is the height of the ball greater than 160 ft?

**(d)** After how many seconds will the ball hit the ground?

### ALGEBRAIC SOLUTION

**(a)** Use the projectile height function.

$$s(t) = -16t^2 + v_0t + s_0 \qquad \text{Let } v_0 = 80 \text{ and}$$

$$s(t) = -16t^2 + 80t + 100 \qquad s_0 = 100.$$

**(b)** The coefficient of $t^2$ is $-16$, so the graph of the projectile function is a parabola that opens down. Find the coordinates of the vertex to determine the maximum height and when it occurs. Let $a = -16$ and $b = 80$ in the vertex formula.

$$t = -\frac{b}{2a} = -\frac{80}{2(-16)} = 2.5$$

$$s(t) = -16t^2 + 80t + 100$$

$$s(2.5) = -16(2.5)^2 + 80(2.5) + 100$$

$$s(2.5) = 200$$

Therefore, after 2.5 sec the ball reaches its maximum height of 200 ft.

### GRAPHING CALCULATOR SOLUTION

**(a)** Use the projectile height function as in the algebraic solution, with $v_0 = 80$ and $s_0 = 100$.

$$s(t) = -16t^2 + 80t + 100$$

**(b)** Using the capabilities of a calculator, we see in **Figure 11** that the vertex coordinates are indeed $(2.5, 200)$.

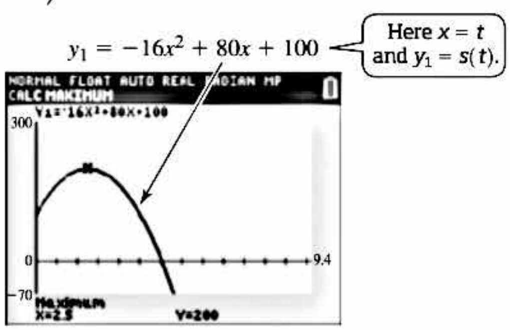



**Figure 11**

***Be careful not to misinterpret the graph in* Figure 11*. It does not show the path followed by the ball. It defines height as a function of time.***