We can now substitute in (3.10.2) $H(j\omega)e^{j\omega t}$ for $y(t)$, $\mathbf{X}(j\omega)e^{j\omega t}$ for $\mathbf{x}(t)$, and $e^{j\omega t}$ for $u(t)$. Then,

$$H(j\omega)e^{j\omega t} = \mathbf{C}\mathbf{X}(j\omega)e^{j\omega t} + \mathbf{D}e^{j\omega t}$$

Substitute (3.10.3) for $\mathbf{X}(j\omega)$ and cancel the terms $e^{j\omega t}$. We then have the desired expression for $H(j\omega)$.

$$H(j\omega) = \mathbf{C}(\mathbf{I}j\omega - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \tag{3.10.4}$$

Note the similarity between this formula and the corresponding expression (2.13.7) for discrete-time systems. The only difference is that $j\omega$ replaces $e^{j\theta}$. In continuous-time systems we traverse the $j\omega$ axis to calculate frequency response. In discrete-time systems, we traverse the unit circle $e^{j\theta}$ to calculate frequency response. This discussion will become more apparent when we consider frequency response in the transform domain.

3.11 SUMMARY

We have discussed three time-domain models for continuous-time systems in this chapter. Our discussion is very similar to that in Chapter 2. In fact, our thought processes are identical. Only the mathematical details are changed. We have used the idea of frequency response as a means of linking the three models together. Frequency-response analysis assumes a linear, constant-parameter system. If the system is time varying, then frequency response as discussed here is no longer a valid method of characterizing the system.

The remaining chapters will take up the concepts involved in describing systems in the transform domain. We shall again consider discrete-time systems first. These transform domain characterizations are useful. They are, like the difference or differential equations and the impulse-response models, input–output characterizations of a system. They do not specify the internal structure of a system. However, they do introduce the idea of poles and zeros of a system. The concept of poles and zeros can be used to give a geometric interpretation of a system's response.

PROBLEMS

3.1. Solve the following differential equations using the given initial conditions.

(a) $(D^2 + 3D + 2)y(t) = 0$ $\quad y(0) = 1, \left.\frac{dy(t)}{dt}\right|_{t=0} = 0$

(b) $(D^2 + 2D + 1)y(t) = 0$ $\quad y(0) = 1, \left.\frac{dy(t)}{dt}\right|_{t=0} = 0$

(c) $(D^3 + 4D^2 + 5D + 2)y(t) = 0 \qquad y(0) = 1, \left.\frac{dy(t)}{dt}\right|_{t=0} = \left.\frac{d^2y(t)}{dt^2}\right. = 0$

(d) $(D^2 - 1)y(t) = 0 \qquad y(0) = 1, \left.\frac{dy(t)}{dt}\right|_{t=0} = 1$

(e) $(D^2 + 1)y(t) = 0 \qquad y(0) = 1, \left.\frac{dy(t)}{dt}\right|_{t=0} = 0$

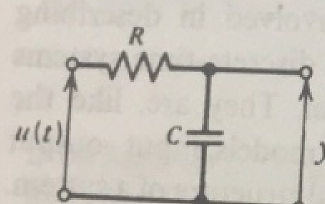
(f) $(D^2 + 1)y(t) = 0 \qquad y(0) = 0, \left.\frac{dy(t)}{dt}\right|_{t=0} = 1$

(g) $(D^2 + 1)y(t) = 0 \qquad y(0) = \left.\frac{dy(t)}{dt}\right|_{t=0} = 1$

(h) $(D^4 - 1)y(t) = 0 \qquad y(0) = 1,$

$$\left.\frac{dy(t)}{dt}\right|_{t=0} = \left.\frac{d^2y(t)}{dt^2}\right|_{t=0} = \left.\frac{d^3y(t)}{dt^2}\right|_{t=0} = 0$$

3.2. In the *RC* circuit shown below, the initial voltage on the capacitor is 2V, that is, $y(0) = 2$. If the input voltage $u(t) = 0$ for $t \geqslant 0$ (obtained by placing a short across the input), what is the output voltage $y(t)$ for $t \geqslant 0$?
Answer: $y(t) = 2e^{-t/RC}$



3.3. Find the appropriate annihilator for each of the following forcing functions.

(a) Ae^{at}

(b) Bt^2e^{at}

(c) $A \sin \omega t$

(d) $A \sin \omega t + B \cos \omega t$

(e) $A \sin (\omega t + \phi)$

(f) $Ae^{at} + Be^{bt} + Ce^{ct}$

(g) $t^3 + B \sin t$

3.4. (a) Time-invariant linear systems forced by sinusoidal inputs form an important class of problems. One model of such systems is made by means of a linear differential equation. Show that the forced solution of the following differential equation must be of the form αe^{kt}:

$$\frac{d^2y(t)}{dt^2} + a\frac{dy(t)}{dt} + by(t) = ce^{kt}$$

where a, b, c, and k are known constants and k is not a root of the characteristic equation.

(b) In view of the result of part (a), what must the form of the forced solution be for a forcing function $c \sin \omega t$ (rather than ce^{kt})?

(c) What is the form of the forced solution if the forcing function is ce^{kt} and k satisfies the equation $k^2 + ak + b = 0$? What physical interpretation can you give to this result?

3.5. Solve the following differential equations.

(a) $(D^4 + 8D^2 + 16)y(t)] = -\sin t$

Answer: $y(t) = c_1 \cos 2t + c_2 \sin 2t + c_3 t \cos 2t + c_4 t \sin 2t - \dfrac{\sin t}{9}$

(b) $(D^3 - 2D^2 + D - 2)[y(t)] = 0, \quad y(0) = \left.\dfrac{dy(t)}{dt}\right|_{t=0} = \left.\dfrac{d^2y(t)}{dt^2}\right|_{t=0} = 1$

Answer: $y(t) = \dfrac{1}{5}(2e^{2t} + 3\cos t + \sin t)$

(c) $(D^4 - D)[y(t)] = t^2$

Answer: $y(t) = c_1 + c_2 e^t + e^{-t/2}\left(c_3 \cos \dfrac{\sqrt{3}t}{2} + c_4 \dfrac{\sin \sqrt{3}t}{2}\right) - \dfrac{t^3}{3}$

3.6. Solve the following differential equations.

(a) $(D^2 + 3D + 2)y(t) = 0 \qquad y(0) = \left.\dfrac{dy(t)}{dt}\right|_{t=0} = 1$

(b) $(D^2 + 3D + 2)y(t) = e^{-t} \qquad y(0) = \left.\dfrac{dy(t)}{dt}\right|_{t=0} = 0$

(c) $(D^2 + 3D + 2)y(t) = e^{-t} \qquad y(0) = \left.\dfrac{dy(t)}{dt}\right|_{t=0} = 1$

Comparing the three solutions, can you generalize the results you obtained? Compare with the solutions to (e), (f), and (g) in Problem 3.1.

3.7. Solve the following differential equations.

(a) $(D^2 + 2D + 1)y(t) = e^{-t} \qquad y(0) = 1, \left.\dfrac{dy(t))}{dt}\right|_{t=0} = 0$

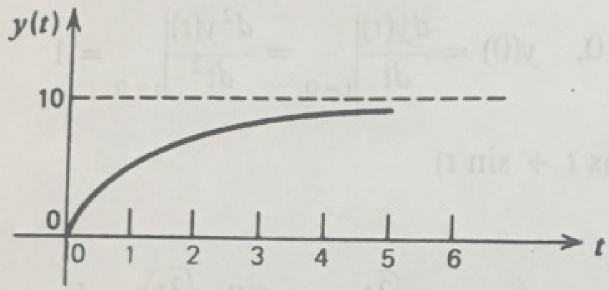
(b) $(D^2 - 1)y(t) = e^{-t} \qquad y(0) = \left.\dfrac{dy(t)}{dt}\right|_{t=0} = 0$

(c) $(D^2 + 1)y(t) = \cos t \qquad y(0) = \left.\dfrac{dy(t)}{dt}\right|_{t=0} = 0$

3.8. Find the impulse response for the continuous-time systems defined by the following differential equations. Verify your solutions by substitution.

(a) $(D^2 + 7D + 12)[y(t)] = u(t)$
(b) $(D^2 + 6D + 9)[y(t)] = u(t)$
(c) $(D^2 + 2D + 9)[y(t)] = u(t)$
(d) $(D^3 + 6D^2 + 12D + 8)[y(t)] = u(t)$
(e) $(D^3 + 4D^2 + 12D + 8)[y(t)] = (D - 1)[u(t)]$

3.9. A system has the following response to a step input. What would be the response of this system to a ramp input, $u(t) = t\xi(t)$? (*Hint*: Assume that the system can be modeled by the use of a first-order differential equation.)



3.10. Derive a model for nanoplankton respiration that accounts for gross community photosynthesis, storage, and respiration.† The following analogies may be helpful.

The input to the system is sunlight, which can be represented by a voltage, say e_b. The production rate p of material by photosynthesis is proportional to the difference in input sunlight and the material already in the system (a "back potential" e_m). The constant of proportionality between production rate p and this difference potential is a conductance G_b. The community respiration rate p_r is assumed to be proportional to e_m. The storage rate p_c of material in the system is assumed to be proportional to the rate of change of material already in the system, and the constant of proportionality is C^{-1}, representative of the storage capacity of the system. In symbols, $p_c = (1/C)de_m/dt$. Also, the total production rate p is equal to the sum of the respiration and storage rates. (Notice that production rates are analogous to currents.)

(a) Obtain a block diagram representation of the system.
(b) Obtain an electrical schematic model for this system.
(c) Using your model, find the dependence of the respiration rate p_r upon a step input of sunlight.

†H. T. Odum et al., "Consequences of Small Storage Capacity in Nonoplankton Pertinent to Primary Production in Tropical Waters," *Journal of Marine Research* (June 1963).

3.11. Given $d^2y(t)/dt^2 + 5[dy(t)/dt] + 6y(t) = e^{-t}$, find $y(0)$ and $(dy/dt)(t)|_{t=0}$ such that the solution is $y(t) = ce^{-t}$. Evaluate c.

3.12. In our treatment of convolution, we assumed that our systems were initially relaxed, with no initial energy storage. How can this method be extended to cover systems in which there is some initial energy storage? Specifically, use the impulse-response function to express the output of a system described by the equation

$$(D^2 + 1)[y(t)] = u(t), \qquad t > 0$$

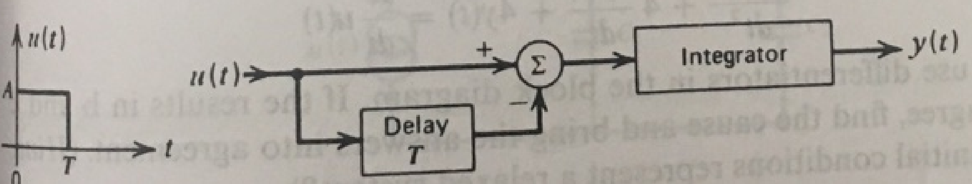
where $y(0) = 1$ and $y^{(1)}(0) = 0$. Notice that initial conditions are non-zero, denoting some initial energy storage in the system.

Answer: $y(t) = \cos t + \int_0^t \sin(t - \tau)u(\tau)\, d\tau$

3.13. Evaluate the following convolutions.

(a) $\xi(t) * \xi(t)$
(b) $\xi(t) * e^{at}\xi(t)$
(c) $t\xi(t) * e^{at}\xi(t)$
(d) $e^{at}\xi(t) * e^{at}\xi(t)$
(e) $e^{at}\xi(t) * e^{-at}\xi(-t)$
(f) $\sin t\xi(t) * \sin t\xi(t)$

3.14. (a) Using the convolution integral, find the output signal for the system shown. This system is often used to smooth a sequence $\{u(nT)\}$ into a continuous-time function. It is called a zero-order hold circuit.



(b) What is the output if $u(t)$ is applied to two of the above systems in cascade?

3.15. Solve the following equations using both the direct method and the convolution method.

(a) $(D^2 + 7D + 12)y(t) = e^t\xi(t), \qquad y(0) = \left.\dfrac{dy(t)}{dt}\right|_{t=0} = 0$

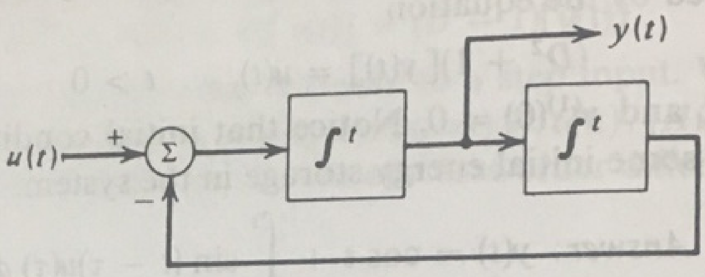
(b) $(D^2 + 3D + 2)y(t) = e^{-t}\xi(t), \qquad y(0) = \left.\dfrac{dy(t)}{dt}\right|_{t=0} = 0$

3.16. Given a system with input $u(t)$ and output $y(t)$ such that

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + \frac{9}{4}y(t) = u(t)$$

find and sketch the corresponding impulse response $h(t)$.

3.17. Find the impulse response **h** for the system shown:



3.18. Given a system with input $u(t)$ and output $y(t)$ related by

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = u(t)$$

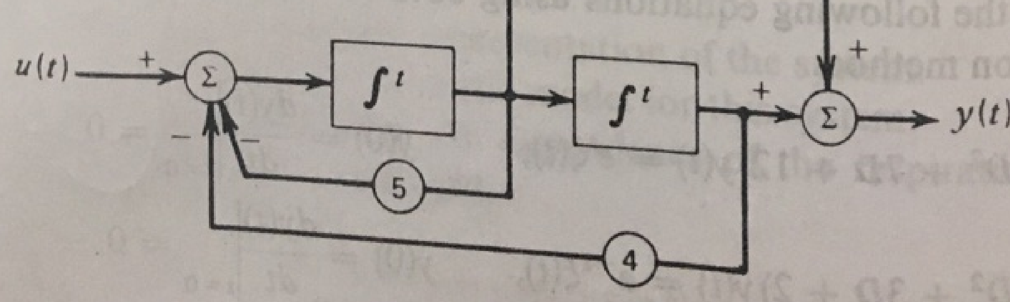
(a) Show a block diagram of the system using integrators, summers, and coefficient multipliers.
(b) Solve for $y(t)$ if $u(t) = e^{-t}\xi(t)$ with $y(0) = dy(t)/dt|_{t=0} = 0$
(c) Find the system impulse response $h(t)$.
(d) Repeat (b) using convolution.

3.19. Repeat steps (a) through (d) of Problem 3.18 for a system described by the input–output relation

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = \frac{d}{dt}u(t)$$

Do not use differentiators in the block diagram. If the results in b and c do not agree, find the cause and bring the answers into agreement. (*Hint:* Do the initial conditions represent a relaxed system?)

3.20. For the system shown below, find a differential equation that relates $y(t)$ and $u(t)$. Solve for $y(t)$ when $u(t) = \sin \omega t\ \xi(t)$ with a relaxed system at $t = 0$.



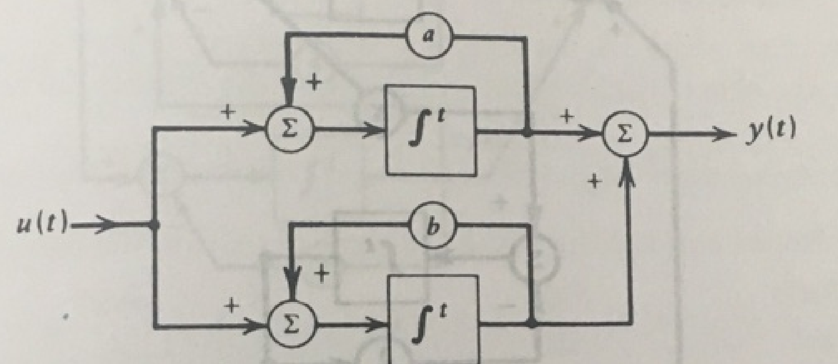
3.21. Can you find a simpler equivalent block diagram for the above system? What is the impulse response of the system? Using convolution, repeat your solution for the output $y(t)$ which results from $u(t) = \sin \omega t \, \xi(t)$.

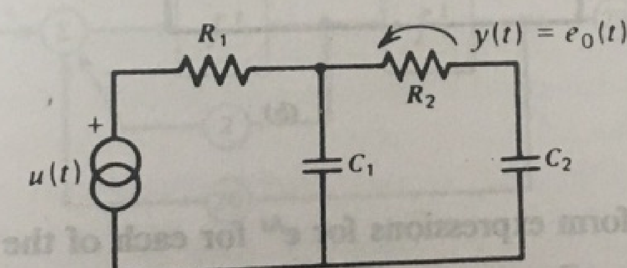
3.22. Determine a system block diagram for which the input $u(t)$ and the output $y(t)$ satisfy the differential equation

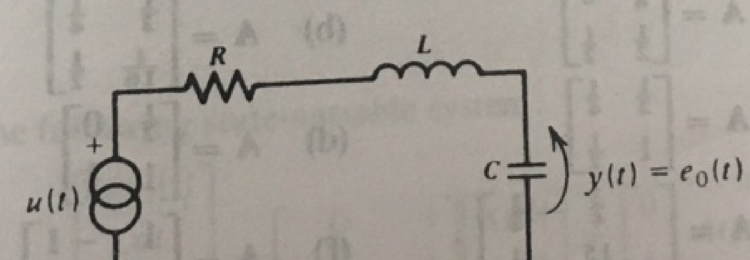
$$\frac{d^2y(t)}{dt^2} + \frac{2dy(t)}{dt} + 2y(t) = \frac{du(t)}{dt}$$

What is the solution to this equation when the input is $u(t) = \cos(\omega_1 t) - \sin(\omega_2 t)$?

3.23. Write state-variable equations for each of the systems shown.

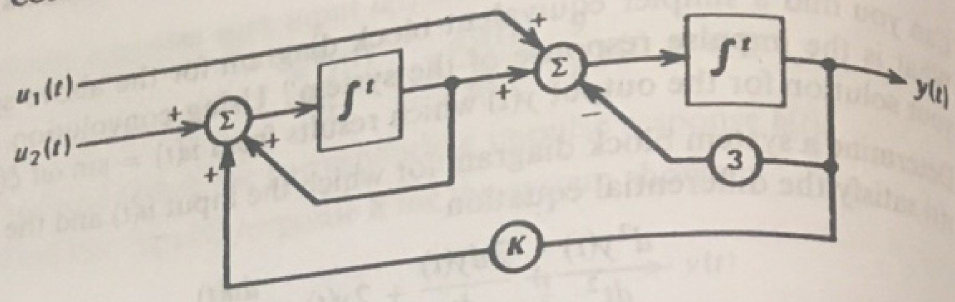




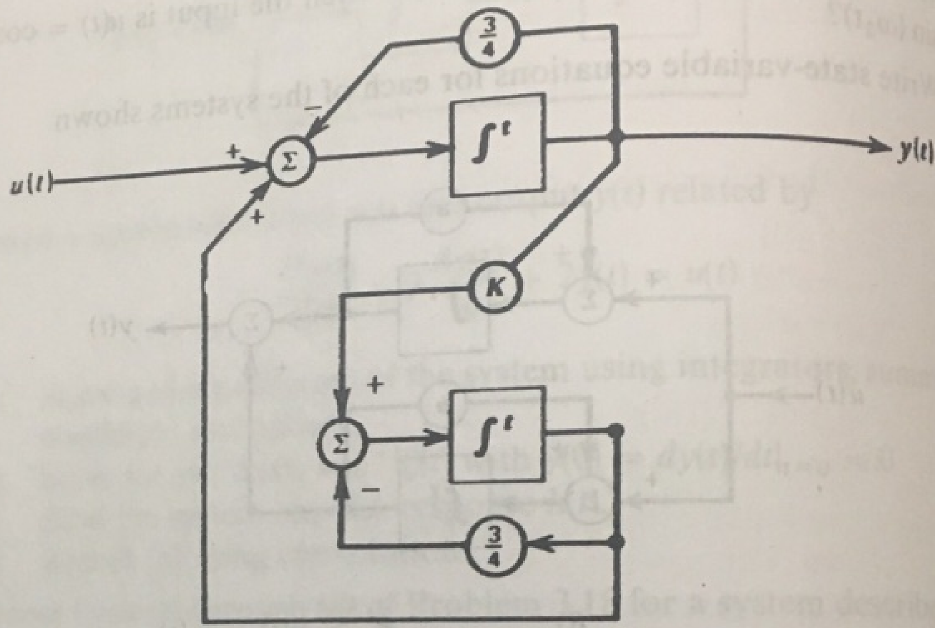


3.24. For what values of the parameters involved is each of the systems above stable?

3.25. Write a state-variable description of the following systems. For what values of K will the systems be stable?



(a)



(b)

3.26. Find closed-form expressions for e^{At} for each of the following matrices

(a) $A = \begin{bmatrix} \frac{3}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ (b) $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{16} & \frac{1}{2} \end{bmatrix}$

(c) $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 1 & \frac{1}{2} \end{bmatrix}$ (d) $A = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & \frac{1}{2} \end{bmatrix}$

(e) $A = \begin{bmatrix} \frac{3}{4} & -\frac{1}{2} \\ -\frac{15}{32} & \frac{1}{2} \end{bmatrix}$ (f) $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

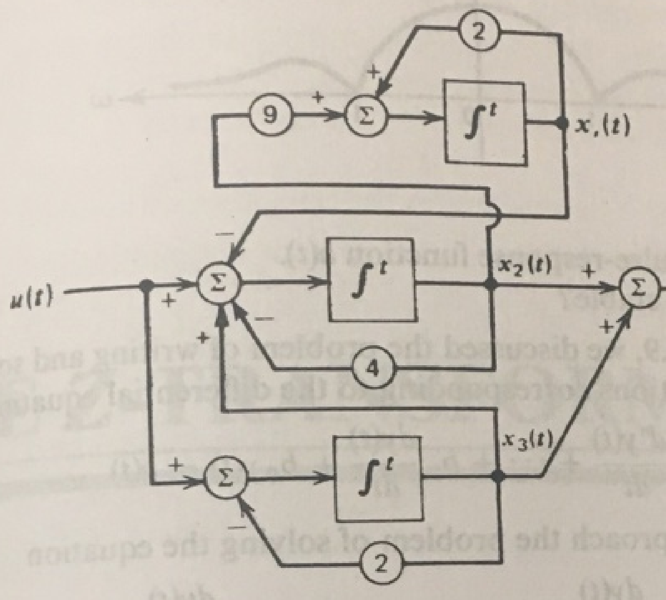
(g) $A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ (h) $A = \begin{bmatrix} -4 & -1 \\ 16 & 4 \end{bmatrix}$

3.27. For the block diagram systems shown below, find

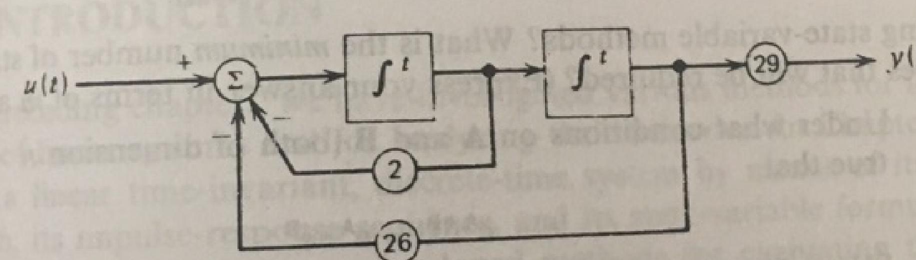
(a) The matrices (**A**, **B**, **C**, **D**) of the state-variable description.

(b) The matrix e^{At}.

(c) The matrix $(j\omega\mathbf{I} - \mathbf{A})^{-1}$.
(d) The frequency-response function, with a sketch of the amplitude and phase responses.
(e) The impulse-response function, with a sketch.



(a)



(b)

3.28. Consider the following state-variable system:

$$\begin{bmatrix} \dfrac{dx_1(t)}{dt} \\ \dfrac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [d]u(t)$$

(a) Find the matrix $(j\omega\mathbf{I} - \mathbf{A})^{-1}$.
(b) Find the matrix $e^{\mathbf{A}t}$.