Reliability Engineering and System Safety 150 (2016) 35–43

Contents lists available at ScienceDirect

Reliability Engineering and System Safety

journal homepage: www.elsevier.com/locate/ress

Reliability evaluation of linear multi-state consecutively-connected systems constrained by m consecutive and n total gaps

Huan Yu [a], Jun Yang [a,*], Rui Peng [b], Yu Zhao [a]

[a] School of Reliability and Systems Engineering, Beihang University, Beijing 100191, China
[b] Donlinks School of Economics and Management, University of Science and Technology Beijing, Beijing 100083, China

ARTICLE INFO

Article history:
Received 25 November 2014
Received in revised form
31 December 2015
Accepted 8 January 2016
Available online 4 February 2016

Keywords:
Linear multi-state consecutive-connected system
Multi-state connection elements
Non-uniform deployment
Universal generating function
Two different types of failures
Genetic algorithm

ABSTRACT

This paper extends the linear multi-state consecutively-connected system (LMCCS) to the case of LMCCS-MN, where MN denotes the dual constraints of m consecutive gaps and n total gaps. All the nodes are distributed along a line and form a sequence. The distances between the adjacent nodes are usually non-uniform. The nodes except the last one can contain statistically independent multi-state connection elements (MCEs). Each MCE can provide a connection between the node at which it is located and the next nodes along the sequence. The LMCCS-MN fails if it meets either of the two constraints. The universal generating function technique is adopted to evaluate the system reliability. The optimal allocations of LMCCS-MN with two different types of failures are solved by genetic algorithm. Finally, two examples are given for the demonstration of the proposed model.

1. Introduction

In modern satellite communications, the purpose is to send information from one point on earth to another through the satellites in space. The information transmission can be simplified to three procedures. First, a radio in one place on earth emits signals up into space. Then one satellite gathers the signals and retransmits them to other satellites along the communication link. Finally, a remote satellite gets the signals and sends them back down to another spot on earth. The whole satellite communications can be modeled as linear multi-state consecutively-connected systems (LMCCSs). The modern satellite communication plays an increasingly important role in the world economy. Thus, researches on the reliability analysis and optimal allocation for the LMCCS would be meaningful and challenging. Generally, the LMCCS can be used to model many systems, such as the fluid transportation systems, wireless communication systems, sensor systems and logistics systems. The $N+1$ orderly connected nodes are evenly distributed along a line and form a sequence, which forms an LMCCS [1]. The distance between two adjacent nodes is defined as a basic length unit [2]. All the nodes can contain statistically independent multi-state connection elements (MCEs) except the last one. Each MCE j can provide a connection between the node at which it is located and the next nodes X_j along the sequence. The X_j is a positive random variable (*r.v.*) and its probability mass function (PMF) is usually given. Any MCE failure may cause a disconnection of the sequence. In other words, the system fails, if the disconnection of any node exists in the sequence.

The LMCCS model was first proposed in Ref. [1]. Algorithms for its reliability evaluation were studied in Refs. [3,4]. The LMCCS model has a source node and a sink node. It can be generalized to the multistate-node acyclic network model which has a source node and a number of sink nodes [5]. The reliability evaluation of LMCCS with fixed retransmission delays was studied in [6]. The expenses of bypass transportation systems (BTS) in LMCCS are introduced in Ref. [7], where the expenses depend on the number of the gaps and the length of the gaps. The reliability analysis for the BTS (such as oil pipeline transportation system and railway network system) can be referred in Refs. [8–13].

The optimal allocation problem of multi-state elements in LMCCS is introduced by Malinowski and Preuss in Ref. [14]. The optimal allocation elements in LMCCS with the common cause failures are studied in Ref. [15]. Then, a special case of LMCCS is proposed by Levitin [16], in which several multi-state elements

Abbreviations: MCEs, multi-state connection elements; PM-LCCS, phased-mission linear consecutively-connected systems; mGCCS, m consecutive gaps linear multi-state consecutively-connected system; LMCCS, linear multi-state consecutively-connected system; BTS, bypass transportation systems; *r.v.*, random variable; LMCCS-N, linear multi-state consecutively-connected system with the constraint of n total gaps; LMCCS-MN, linear multi-state consecutively-connected system subject to the dual constraints of m consecutive gaps and n total gaps; GA, genetic algorithm; UGF, universal generating function; PMF, probability mass function.

* Corresponding author. Tel./fax: +86 10 82316003.
E-mail address: tomyj2001@buaa.edu.cn (J. Yang).

http://dx.doi.org/10.1016/j.ress.2016.01.010
0951-8320/

Nomenclature

E_j	set of consecutively ordered nodes $\{1,...,j\}$
T_j	the most remote location with which the nodes in the set E_j can be connected
G_j	the most remote location with which the node j can be connected directly
$g_{j,k}$	k-th realization of $r.v.$ G_j
$p_{j,k}$	probability of the realization $r.v.$ $g_{j,k}$
$\boldsymbol{G}_j$	vector of different realization of G_j
$\boldsymbol{P}_j$	vector of probabilities of random realization of G_j
$1(A)$	having the value 1 when A is true and the value 0 when A is false

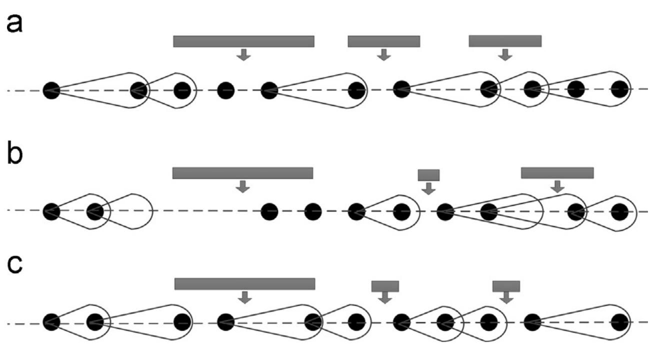

Fig. 1. Linear multistate connected sensors in working state (a), and failed state (b,c).

can be allocated in the same node while some nodes can remain empty. When several elements are allocated in a node, they can be destroyed by common cause failures with the node suffering attack. The optimal maintenance and allocation of elements in LMCCS were solved by Peng et al. in Ref. [17], which is constrained with the system availability requirement. Then, the LMCCS model is extended to the phased-mission linear consecutively-connected systems (PM-LCCS) in Ref. [18]. In each phase, the PM-LCCS can be reduced to an LMCCS model and the nodes are in equidistant locations. The reliability analysis for the phased-mission system is studied in Refs. [19–21]. The PM-LCCS model with common cause failures is studied in Ref. [22].

The LMCCS model was further extended to m consecutive gaps linear multi-state consecutively-connected system (mGCCS) [23]. A gap exists in the sequence if a node cannot be connected with any previous node. The m consecutive gaps in the sequence mean that at least m consecutive nodes cannot be connected with any previous node. The structure of the mGCCS model is the same as that of the traditional LMCCS model. The nodes in the mGCCS model is in equidistant locations along a line, and the system fails as long as there is at least m consecutive gaps. When $m=1$, no gap is allowed, and the mGCCS model reduces to a traditional LMCCS model. The condition of the m consecutive gaps is the only constraint for the mGCCS model. Another generalization of the LMCCS model is the LMCCS with the constraint of n total gaps (LMCCS-N) [2]. The LMCCS-N fails if it contains at least n nodes disconnected with any previous node. In other words, the LMCCS-N fails if the total number of gaps reaches a specified limit n. The structure of the LMCCS-N model is the same as that of the traditional LMCCS model. The nodes in the LMCCS-N model is also in equidistant locations along a line. The traditional LMCCS model is a special case of the LMCCS-N model with $n=1$. Since the LMCCS-N model is only focused on the constraint of the total number of gaps, it is not suitable for the LMCCS with the constraint of consecutive gaps. Another extended model of LMCCS is the LMCCS with the combined gap constraints (m/nCCS) [24]. The m/nCCS model fails if there exist at least m nodes not connected with any previous node or at least n consecutive nodes not connected with any previous node. The structure of the m/nCCS model is the same as that of the traditional LMCCS model. The nodes in the m/nCCS model is also in equidistant locations along a line. In the traditional LMCCS model [1,4,15,16] and its extended models: LMCCS-N [2], mGCCS [23], and m/nCCS [24], the structures are the same and the distances between the adjacent nodes are equal to a basic length unit, which may be not true in many practical applications.

Take the linear wireless sensor networks for example, sensors are usually allocated to the non-uniform distributed nodes along a line. The structure is shown in Fig. 1. Each sensor can cover different zones, which is dependent on the internal operation modes and the external environment factors, and the size of covered zones can vary discretely over different sensors. They must detect some objects that traverse the line. Usually, there are two kinds of objects: small objects and large objects. If a gap exists along the line, some small objects can traverse the line, and cannot be detected. If the number of consecutive gaps exceeds the size of a large object, the large object can traverse the line, and otherwise it cannot pass the line. When the objects pass the line, they cannot be detected, and then the system may not work. The system consists of two different types of failures: one is subject to the number of undetected objects (as shown in Fig. 1(c)), and the other is due to a large object being undetected (as shown in Fig. 1(b)). That is to say, when the number of undetected objects exceeds a pre-specified value by the engineering requirement, the system fails. Moreover, when a large object passes the line, it will be undetected, and then the system fails. When either of the two different types of failures happens, regardless of which one happens first, the system fails. In other words, the system survives if neither of the two different types of failures occurs (as shown in Fig. 1(a)).

The traditional LMCCS model and its extended models: LMCCS-N [2], mGCCS [23], and m/nCCS [24], can also not be applied directly to evaluate the crude oil transportation system with the unevenly distributed nodes. A crude oil transportation system is a linear long distance transport system. It supplies crude oil from oil well to a refinery, which is composed of crude oil transportation pipelines and crude oil pumps. Oil pumps are located at the consecutive nodes between the pipelines. They provide enough pressure to transfer the crude oil. The distances between any two adjacent nodes are usually not equal because of the external environment factors. Fig. 2 shows a crude oil transportation system. Oil pumps are the key instruments in the crude oil transportation system. In certain degree, the crude oil transportation pipelines remain intact. If there exist nodes that are not connected with any previous node, the oil cannot be transferred to these nodes by the system and the dynamic bypass transfer instruments, such as oil truck and tank truck, should be applied. With the constraints, such as transportation capacity, energy consumption and greenhouse gas emissions, the bypass instruments cannot be competent for too many consecutive gaps. Subject to budget cost constraint, the amount of the bypass instruments is also limited, which limits the total number of the gaps. Either when the number of consecutive gaps exceeds one pre-specified value, or when the number of total gaps exceeds some pre-specified value, the system fails.

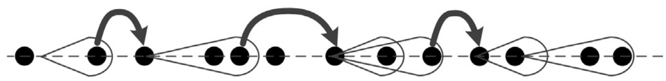

Fig. 2. Crude oil transportation system.

Motivated by the real applications, we proposed a new model of linear multi-state consecutively-connected system subject to the dual constraints of m consecutive gaps and n total gaps (LMCCS-MN). Its structure is the same as the traditional LMCCS model, but it adopts the uneven deployment of nodes. The LMCCS-MN fails if it meets either of the two constraints.

Considering two different types of failures, the proposed LMCCS-MN model is extended with previous researches [2,23,24]. Obviously, the traditional LMCCS model, the LMCCS-N model, the mGCCS model and the m/nCCS model are four different special cases of the proposed LMCCS-MN model in this paper. The limitation of the four different special cases is that they can only deal with the LMCCS with the evenly distributed nodes. The proposed LMCCS-MN model overcomes the limitation and involves the unevenly distributed nodes in the LMCCS.

When $m=1$ or $n=1$, no gaps are allowed, and the LMCCS-MN model can reduce to the LMCCS model if it adopts the evenly distributed nodes.

When $m \geq n$, the constraint of m consecutive gaps is in vain, only the constraint of n total gaps is effective and the LMCCS-MN model can reduce to LMCCS-N model if it adopts the evenly distributed nodes.

Since the reliability of LMCCS-MN is considerably influenced by the element positions, when $n=N$, the constraint of n total gaps is no effect, only the constraint of m consecutive gaps is effective, and the LMCCS-MN model can reduce to the mGCCS model if it adopts the evenly distributed nodes.

When $1<m<n<N$, the LMCCS-MN model is subject to the dual constraints. The proposed model introduces the uneven deployment of nodes. If it has uniform deployment of nodes, it will reduce to the m/nCCS model.

In this paper, the universal generating function (UGF) technique is adopted to evaluate the reliability of the proposed model. The UGF technique has been proved an efficient method in multi-state system (MSS) reliability analysis [25–31]. The algorithm for system reliability evaluation is based on the Refs. [2,23,24]. In the evaluation algorithm [23] for the mGCCS model, a random variable is used to count the consecutive gaps. In the evaluation algorithm [2] for the LMCCS-N model, an integer counter is incorporated into the UGF to keep the number of gaps. In the m/nCCS model [24], a function is used to map the number of consecutive gaps and the total number of gaps into an integer number. In the evaluation algorithm for the proposed LMCCS-MN model, a random vector clearly and concisely records both the number of consecutive gaps and the number of total gaps. Besides the two different types of failures, the non-uniform deployment of nodes is involved in the proposed LMCCS-MN model. Anyway, the improvement for the proposed LMCCS-MN model will increase the algorithm complexity and the amount of calculation. In the LMCCS-MN model, the locations for different nodes are determined in advance by the external environment factors and cannot be changed. Fortunately, the MCEs can be allocated to different nodes. Different allocation strategies of the MCEs should be compared in order to improve the reliability of the LMCCS-MN model. The problem of optimal allocation of MCEs in the sequence is solved by genetic algorithm (GA) technique.

The remainder of the paper is organized as follows. Section 2 presents the model description of the LMCCS-MN model, an algorithm of its reliability evaluation is proposed based on UGF in Section 3, and the computational complexity of the proposed reliability evaluation algorithm is discussed in Section 4. Two examples are given in Section 5 to demonstrate the reliability evaluation and optimal allocation of the proposed model. Finally, some conclusions are given in Section 6.

2. Model descriptions

The LMCCS-MN is subject to the dual constraints of m consecutive gaps and n total gaps. It experiences two different types of failures: failure of consecutive gaps and failure of total gaps. That is to say, when the number of consecutive gaps exceeds the pre-specified value M, the failure of consecutive gaps occurs, and then the system fails. Furthermore, when the number of total gaps exceeds the pre-specified value n, the failure of total gaps occurs, and then the system fails. When either of the two different types of failures happens, regardless of which one happens first, the system fails. The two different failures are s-dependent. They can occur simultaneously. If the system generates more than m consecutive gaps and more than n gaps in total at the same time, then the two different failures occur simultaneously.

The LMCCS-MN is composed of $N+1$ orderly connected nodes. All the nodes, which can contain MCEs except the last one, are allocated along a line. The distance between node $j \in \{1, \cdots, N+1\}$ and the first node is referred as S_j, specially $S_1=0$. The distances between the adjacent nodes are usually unequal. The unit of distance is usually given such as miles, feet, kilometers or meters according to the need of engineering practice. The range of the MCE in the node $j \in \{1, \cdots, N\}$ can be from 0 to X_j along the line. All the nodes in the range of the MCE can be connected directly. Then the most remote location, with which the node j can be connected directly, can be represented by the positive $r.v.$ $G_j = \min(S_j+X_j, S_{N+1})$. It is allowed that some node does not contain any MCE. If the node j does not contain any MCE, the case can be seen as a disabled MCE contained with the single state $X_j \equiv 0$. In this case, the most remote location connected directly with the node j, is the location that the node j itself is allocated, which gives $G_j \equiv S_j$.

The node $j \in \{1, \ldots, N\}$ can be directly connected with K_j different locations, which can be represented by the positive $r.v.$ G_j. The vector $\boldsymbol{G}_j = g_{j,1}, \cdots, g_{j,K_j}$ stands for the vector of different states corresponding to the node j, where $g_{j,k}$ is the k-th realization of $r.v.$ G_j. In the vector, the different states are sorted in ascending order. The vector $\boldsymbol{P}_j = p_{j,1}, \ldots, p_{j,K_j}$ stands for the vector of probabilities of random realization of G_j, where $p_{j,k}$ is the probability of the $r.v.$ G_j takes the realization $g_{j,k}$, and $\sum_{k=0}^{K_j} p_{j,k} = 1$.

Let set E_j stands for the set of consecutive nodes $\{1, \ldots, j\}$. The positive $r.v.$ T_j represents the most remote location with which the nodes in the set E_j can be connected. It is equivalent to the maximum value of G_i, $i \in \{1, \ldots, j\}$. It can be written as $T_j = \max_{1 \leq i \leq j}\{G_i\}$. T_j can be obtained by the recursion formula $T_j = \max\{T_{j-1}, G_i\}$ for any $j \geq 1$, where the initial value is $T_0 = 0$ by definition. If $r.v.$ $T_j < S_{j+1}$, node $j+1$ is connected with none of the previous nodes in the set E_j, and the gap exists.

The failure of consecutive gaps occurs when none of the nodes, in the m consecutive nodes $\{s, s+1, \ldots, s+m-1\}$, can connect with any previous node. It can be written as

$$\prod_{j=s-1}^{m+s-2} 1\left(T_j < S_{j+1}\right) = 1. \tag{1}$$

The $N+1$ nodes in the system can be divided into $N-m+1$ sets of m consecutive nodes, which are $\{2, 3, \ldots, m+1\}$, $\{3, 4, \ldots, m+2\}$, $\cdots$, $\{N+1-m, N+2-m, \ldots, N\}$, and $\{N+2-m, N+3-m, \ldots, N+1\}$, respectively. If there exists a set of m consecutive nodes, in which all of the nodes are unable to connect with the previous nodes, then the system fails. Thus, the event that the failure of m consecutive gaps occurs can be represented as

$$\sum_{s=2}^{N+2-m}\left(\prod_{j=s-1}^{m+s-2} 1\left(T_j < S_{j+1}\right)\right) > 0. \tag{2}$$

The failure of total gaps occurs if at least n nodes in the system are unable to connect with any previous node. This kind of failure can be expressed as

$$\sum_{j=1}^{N} 1\left(T_j < S_{j+1}\right) \geq n. \tag{3}$$

Therefore, the system survives when no m consecutive gaps occurs and the total number of gaps is less than n. The system reliability can be defined as

$$R = \Pr\left\{\sum_{s=2}^{N+2-m}\left(\prod_{j=s-1}^{m+s-2} 1\left(T_j < S_{j+1}\right)\right) = 0, \text{and}, \sum_{j=1}^{N} 1\left(T_j < S_{j+1}\right) < n\right\}. \tag{4}$$

It also can be written as

$$\begin{aligned} R &= 1 - \Pr\left\{\sum_{s=2}^{N+2-m}\left(\prod_{j=s-1}^{m+s-2} 1\left(T_j < S_{j+1}\right)\right) > 0, \text{or}, \sum_{j=1}^{N} 1\left(T_j < S_{j+1}\right) \geq n\right\} \\ &= 1 - \Pr\left\{\sum_{s=2}^{N+2-m}\left(\prod_{j=s-1}^{m+s-2} 1\left(T_j < S_{j+1}\right)\right) > 0\right\} \\ &\quad + \Pr\left\{\sum_{s=2}^{N+2-m}\left(\prod_{j=s-1}^{m+s-2} 1\left(T_j < S_{j+1}\right)\right) > 0, \text{and}, \sum_{j=1}^{N} 1\left(T_j < S_{j+1}\right) \geq n\right\} \\ &\quad - \Pr\left\{\sum_{j=1}^{N} 1\left(T_j < S_{j+1}\right) \geq n\right\}. \end{aligned} \tag{5}$$

3. An algorithm for the reliability evaluation

In this paper, we apply UGF technique for the reliability evaluation of LMCCS-MN, since the UGF technique is very effective in reliability analysis and design of multi-state systems. For more detail, it can be referred in [32].

The PMF of $r.v.$ G_j can be defined by a u-function as follows

$$u_j(z) = \sum_{k=1}^{K_j} p_{j,k} z^{g_{j,k}}, \tag{6}$$

where $p_{j,k}$ is the probability of the realization $g_{j,k}$.

The positive $r.v.$ T_j represents the most remote location with which the nodes in the set E_j can be connected. Its recursion formula is $T_j = \max\{T_{j-1}, G_i\}$ for any $j \geq 1$, where the initial value is $T_0 = 0$.

In order to record the realizations of the random number of gaps in LMCCS-MN, we define two different integer $r.v.$ C_j and D_j. The integer $r.v.$ C_j counts the random number of consecutive gaps from the node j to the previous nodes in LMCCS-MN. The counter C_j increments only when $T_j < S_{j+1}$ and is assigned 0 when $T_j \geq S_{j+1}$. The integer $r.v.$ C_j can be obtained by the recursion formula

$$C_j = \begin{cases} 0, \text{ if } T_j \geq S_{j+1} \\ C_{j-1} + 1, \text{if } T_j < S_{j+1} \end{cases}, \tag{7}$$

for any $j \geq 1$, where the initial value is $C_0 = 0$.

The integer $r.v.$ D_j counts the random number of total gaps in the set E_j, where E_j stands for the set of consecutive nodes $\{1, \ldots, j\}$ in LMCCS-MN. The counter D_j increments only when $T_j < S_{j+1}$. In other words, it increments once a new gap exists. The integer $r.v.$ D_j can be obtained by the recursion formula

$$D_j = \begin{cases} D_{j-1}, \text{if } T_j \geq S_{j+1} \\ D_{j-1} + 1, \text{if } T_j < S_{j+1} \end{cases}, \tag{8}$$

for any $j \geq 1$, where the initial value is $D_0 = 0$.

Then, we define a random vector $W_j = (T_j, C_j, D_j)$ of node j. It simultaneously records the random number of the most remote location T_j connected to the nodes belonging to the set E_j, the random number of the consecutive gaps C_j from the node j to the previous nodes, and the random number of total gaps D_j in the set E_j.

Thus, the random vector W_j can be obtained by the recursion formula $W_j = \Omega(W_{j-1}, G_j)$ for any $j \geq 1$, where the initial value is $W_0 = (0, 0, 0)$. From Eqs. (7) and (8), the recursion formula $W_j = \Omega(W_{j-1}, G_j)$ can be defined as follows

$$W_j = \Omega(W_{j-1}, G_j) = \begin{cases} (\max\{T_{j-1}, G_j\}, 0, D_{j-1}), \text{if } T_j \geq S_{j+1} \\ (\max\{T_{j-1}, G_j\}, C_{j-1}+1, D_{j-1}+1), \text{if } T_j < S_{j+1} \end{cases}. \tag{9}$$

The PMF of random vector W_j can be defined by a u-function as follows:

$$U_j(z) = \sum_{h=1}^{H_j} Q_{j,h} z^{w_{j,h}} = \sum_{h=1}^{K_j} Q_{j,h} z^{(t_{j,h}, c_{j,h}, d_{j,h})}, \tag{10}$$

where $Q_{j,h}$ is the probability of the realization of $w_{j,h} = (t_{j,h}, c_{j,h}, d_{j,h})$, for $j = 1, \cdots, N, h = 1, \cdots, H_j$. As $\Pr\{W_0 = (0,0,0)\} = 1$, $U_0(z) = 1z^{(0,0,0)} = z^{(0,0,0)}$. To obtain the u-function of the random vector W_j, we can apply the composition operator with function Ω recursively

$$\begin{aligned} U_j(z) = U_{j-1}(z) \otimes_\Omega u_j(z) &= \sum_{h=1}^{H_{j-1}} \sum_{k=1}^{K_j} Q_{j-1,h} p_{j,k} z^{\Omega(w_{j-1,h}, g_{j,k})} \\ &= \sum_{h=1}^{H_j} Q_{j,h} z^{(t_{j,h}, c_{j,h}, d_{j,h})}, \end{aligned} \tag{11}$$

for $j = 1, \ldots, N$. The initial value is $U_0(z) = 1z^{(0,0,0)} = z^{(0,0,0)}$.

Note that the realization $t_{j,h}$ in the u-function $U_j(z)$ of node j is a positive value, and takes values from the interval $[S_j, S_{N+1}]$. Both the realization of counter $c_{j,h}$ and the realization of counter $d_{j,h}$ are integer values and cannot exceed j. That is no more than j gaps can exist in the set E_j, because E_j contains only j nodes. Meanwhile, it holds $c_{j,h} \leq d_{j,h}$, for $j = 1, \cdots, N$, because the number of consecutive gaps $c_{j,h}$ will never exceed that of total gaps $d_{j,h}$ in the set E_j.

The u-function $U_j(z)$ of node j can contain some terms with the same exponent. The circumstances arise when different combinations of realizations of W_{j-1} and G_j result in the same W_j. These terms are called the like terms and can be collected. For example, the PMF of W_5 represented by u-function is $U_5(z) = q_{5,1}z^{(5,1,2)} + q_{5,2}z^{(6,0,1)} + q_{5,3}z^{(7,0,1)}$. The PMF of G_6 represented by u-function is $u_6(z) = p_{6,1}z^6 + p_{6,2}z^7$. The locations of the nodes are $S_5 = 5$, $S_6 = 6$ and $S_7 = 7$ respectively. After applying the composition operator (11), we get the PMF of W_6 represented by u-function

$$\begin{aligned} U_6(z) = U_5(z) \otimes_\Omega u_6(z) &= (q_{5,1}z^{(5,1,2)} + q_{5,2}z^{(6,0,1)} + q_{5,3}z^{(7,0,1)}) \otimes_\Omega (p_{6,1}z^6 + p_{6,2}z^7) \\ &= q_{5,1}p_{6,1}z^{(6,2,3)} + q_{5,2}p_{6,1}z^{(6,1,2)} + q_{5,3}p_{6,1}z^{(7,0,1)} + q_{5,1}p_{6,2}z^{(7,0,2)} \\ &\quad + q_{5,2}p_{6,2}z^{(7,0,1)} + q_{5,3}p_{6,2}z^{(7,0,1)} = q_{5,1}p_{6,1}z^{(6,2,3)} + q_{5,2}p_{6,1}z^{(6,1,2)} \\ &\quad + (q_{5,3}p_{6,1} + q_{5,2}p_{6,2} + q_{5,3}p_{6,2})z^{(7,0,1)} + q_{5,1}p_{6,2}z^{(7,0,2)}. \end{aligned}$$

The result shows that three different combinations of realizations of W_5 and G_6 produce $W_6 = (7, 0, 1)$. By collecting the like terms in the u-function, we can achieve the simplified u-function. This is an effective way to reduce the computational complexity of further applications of the composition operators over the obtained u-function.

Some terms with $c_{j,h} = m$ in the u-function $U_j(z)$ of node j correspond to the event of failure of m consecutive gaps in the set of nodes E_{j+1}. While, certain terms with $d_{j,h} = n$ in the u-function $U_j(z)$ of node j correspond to the event of failure of n total gaps in the set of nodes E_{j+1}. Moreover, a number of terms with $c_{j,h}=m$ and $d_{j,h}=n$ in the u-function $U_j(z)$ of node j correspond to the joint event of failure of m consecutive gaps and failure of n total gaps in the set of nodes E_{j+1}. To obtained the probability that the set of nodes E_{j+1} contains m consecutive gaps or n total gaps, the following operator can be used over $U_j(z)=\sum_{h=1}^{H_j} Q_{j,h}z^{w_{j,h}}$,

$$\begin{aligned}\Psi_{m,n}(U_j(z)) &= \Psi_{m,n}\left(\sum_{h=1}^{H_j} Q_{j,h}z^{w_{j,h}}\right) = \Psi_{m,n}\left(\sum_{h=1}^{H_j} Q_{j,h}z^{(t_{j,h},c_{j,h},d_{j,h})}\right)\\ &= \sum_{h=1}^{H_j} 1(c_{j,h}=m \text{ or } d_{j,h}=n)Q_{j,h} = \sum_{h=1}^{H_j} 1(c_{j,h}=m)Q_{j,h}\\ &+ \sum_{h=1}^{H_j} 1(d_{j,h}=n)Q_{j,h} - \sum_{h=1}^{H_j} 1(c_{j,h}=m \text{ and } d_{j,h}=n)Q_{j,h}.\end{aligned} \tag{12}$$

where $1(A)$ is an indicator function of A.

The event of system failure is an union of the mutually exclusive failure events $\Lambda_r \bigcup \left(\Lambda_{r+1}\bigcap\overline{\Lambda}_r\right)\bigcup\cdots\bigcup\left(\Lambda_N\bigcap\overline{\Lambda}_{N-1}\right)$, where $r=\min(m,n)$ The event Λ_j is that the set of nodes E_{j+1} contains m consecutive gaps or n total gaps. If the event Λ_j occurs, the system fails, which is independent of the states of nodes not belonging to E_{j+1}. The event $\Lambda_j\bigcap\overline{\Lambda}_{j-1}$ is that the set of nodes E_{j+1} contains m consecutive gaps or n total gaps, whereas the set of nodes E_j contains both less than m consecutive gaps and less than n total gaps. The failure probability of the entire system is

$$F=\Pr\{\Lambda_r\}+\Pr\left\{\Lambda_{r+1}\bigcap\overline{\Lambda}_r\right\}+\ldots+\Pr\left\{\Lambda_N\bigcap\overline{\Lambda}_{N-1}\right\}, \tag{13}$$

where $r=\min(m,n)$.

When $j<r$, the set of nodes E_{j+1} contains neither more than m consecutive gaps nor more than n total gaps from node 2. In other words, when $j<r$, the number of consecutive gaps is always less than m, and the total number of gaps is always less than n. Thus, the event Λ_j cannot occur and $\Pr\{\Lambda_j\}=0$ for any $j<r$.

The probability $\Pr\{\Lambda_r\}=$ Pr{the set of nodes E_{r+1} contains m consecutive gaps or n total gaps} can be obtained by the operator $\Psi_{m,n}(U_r(z))$ in (12), where $r=\min(m,n)$. The probability $\Pr\left\{\Lambda_j\bigcap\overline{\Lambda}_{j-1}\right\}$ is Pr{the set of nodes E_{j+1} contains m consecutive gaps or n total gaps | the set of nodes E_j contains both less than m consecutive gaps and less than n total gaps}. The method of calculation for the probabilities can be referred to the Refs. [2,23]. The probability Pr $\left\{\Lambda_j\bigcap\overline{\Lambda}_{j-1}\right\}$ can be obtained by the following three steps:

Step 1. Delete the terms with $C_{j-1}=m$ (excluding the event that m consecutive gaps exist in the set E_j) and the terms with $D_{j-1}=n$ (excluding the event that n total gaps exist in the set E_j), and the truncated u-function of $U_{j-1}(z)$ can be written as $\tilde{U}_{j-1}(z)$.
Step 2. Apply the operator $\otimes_\Omega$ in (11) over the truncated u-function $\tilde{U}_{j-1}(z)$ and u-function $u_j(z)$, and $U_j(z)=\tilde{U}_{j-1}(z)\otimes_\Omega u_j(z)$ is obtained.
Step 3. Apply the operator $\Psi_{m,n}(U_j(z))$ in (12) over the u-function $U_j(z)$, and the probability $\Pr\left\{\Lambda_j\bigcap\overline{\Lambda}_{j-1}\right\}$ is obtained.
The reliability of LMCCS-MN can be obtained by the following recursive algorithm with four steps, when the PMF of G_j is given for $1\le j\le N$. The algorithm of reliability evaluation for LMCCS-MN is referred to the Refs. [2,23]. It is as follows:
Step 1. Get the u-function of G_j from the $\boldsymbol{G}_j, \boldsymbol{P}_j$ and Eq. (6), for $1\le j\le N$.
Step 2. Assign $U_0(z)=z^{(0,0,0)}, F=0$.
Step 3. For $1,\ldots,N$: apply $U_j(z)=U_{j-1}(z)\otimes_\Omega u_j(z)$, add the value $\Psi_{m,n}(U_j(z))$ to F, delete the terms with $C_j=m$ and the terms with $D_j=n$ from $U_j(z)$, and then collect the like terms.
Step 4. Gain $R=1-F$.

4. Computational complexity of the proposed algorithm

In the algorithm of reliability evaluation, the operator $\otimes_\Omega$ should be applied over u-function $U_{j-1}(z)$ and $u_j(z)$, for $j=1,\ldots,N$. It has been used N times. The computational complexity of each operator is proportional to the product of the numbers of the terms in the u-function $U_{j-1}(z)$ and $u_j(z)$. Then the total computational complexity is proportional to

$$\sum_{j=1}^{N} |U_{j-1}(z)||u_j(z)|. \tag{14}$$

In the most complicated situation, each r.v. G_j can take all possible values from j to $N+1$

$$|u_j(z)|=N-j+2. \tag{15}$$

The r.v. T_j takes values from the interval $[S_j,S_{N+1}]$. Assume that the realizations of T_j are separately in different intervals $[(S_j,S_{j+1}), [(S_{j+1},S_{j+2}),\ldots,[(S_N,S_{N+1}), [S_{N+1},S_{N+1}]$. T_j has no more than $N-j+2$ different realizations. The counter C_j takes no more than $\min(j,m)$ integer values from 0 to $m-1$, while the counter D_j takes no more than $\min(j,n)$ integer values from 0 to $n-1$. This is because when the counter C_j is m or the counter D_j is n, the terms in the u-function $U_{j-1}(z)$ will be removed. Thus,

$$|U_{j-1}(z)|=(N-j+2)\times\min(j,m)\times\min(j,n). \tag{16}$$

The total computational complexity is proportional to

$$\begin{aligned}\sum_{j=1}^{N}|U_{j-1}(z)|\,|u_j(z)| &= \sum_{j=1}^{N}(N-j+2)(N-j+3)\times\min(j-1,m)\\ &\times\min(j-1,n)<\sum_{j=1}^{N}(N-j+2)(N-j+3)\\ &\times mn = mn\frac{(N+1)(N+2)(N+3)}{3}-2mn.\end{aligned} \tag{17}$$

Therefore, according to the Eq. (17), the computational complexity of the proposed algorithm is polynomial, increases with m, n and N. It is not faster than $O\left(mnN^3\right)$, where m is the allowed number of consecutive gaps, n is the allowed total number of the gaps and N is the number of nodes that can host MCEs. The computational complexity of the mGCCS model in Ref. [23] is $O\left(N^3\right)$, while that of the LMCCS-N model in Ref. [2] is $O\left(n^2N^2\right)$. The computational complexity of the proposed algorithm is higher than that of both the mGCCS model and the LMCCS-N model. Since the two different types of failures are involved in the proposed model, which increases the complexity of algorithm and the amount of calculation.

5. Illustrative examples

5.1. Analytical example

Consider an LMCCS-MN of five nodes ($N+1=6$). The locations of the nodes are $S_1=0, S_2=2, S_3=3, S_4=4, S_5=5$, and $S_6=6$. The distance between the first node and the second node is 2, while the other distances between the adjacent nodes all equal to 1. The system fails if either the event of two consecutive gaps ($m=2$) or the event of three total gaps ($n=3$) exists. The PMF of the first

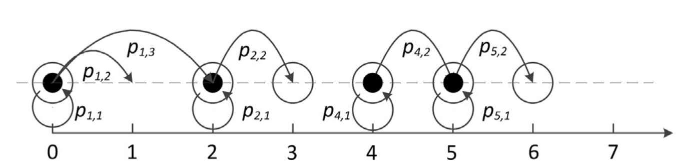

Fig. 3. Analytical example of LMCCS-MN.

five nodes are $\boldsymbol{G}_1=\{0,1,2\}$, $\boldsymbol{P}_1=\{p_{1,1},p_{1,2},p_{1,3}\}$, $\boldsymbol{G}_2=\{2,3\}$, $\boldsymbol{P}_2=\{p_{2,1},p_{2,2}\}$,$\boldsymbol{G}_3=\{3\}$, $\boldsymbol{P}_3=\{1\}$,$\boldsymbol{G}_4=\{4,5\}$, $\boldsymbol{P}_4=\{p_{4,1},p_{4,2}\}$,$\boldsymbol{G}_5=\{5,6\}$, and $\boldsymbol{P}_5=\{p_{5,1},p_{5,2}\}$. The structure is show in Fig. 3.

Following Step 1 of the algorithm of reliability evaluation, the u-functions of the nodes are

$$u_1(z)=p_{1,1}z^0+p_{1,2}z^1+p_{1,3}z^2,$$

$$u_2(z)=p_{2,1}z^2+p_{2,2}z^3,$$

$$u_3(z)=z^3, u_4(z)=p_{4,1}z^4+p_{4,2}z^5,$$

$$u_5(z)=p_{5,1}z^5+p_{5,2}z^6.$$

Then we assign $U_0(z)=z^{(0,0,0)}$,$F=0$, and apply the operator $\otimes_\Omega$ in (11) recursively, $U_1(z),U_2(z),U_3(z),U_4(z),U_5(z)$ and truncated u-functions $\tilde{U}_2(z),\tilde{U}_3(z),\tilde{U}_4(z)$ are obtained, which can be seen in the Appendix in detail. From their relation with F the failure probability of LMCCS-MN is obtained,

$$F=p_{2,1}+p_{2,2}p_{4,1}+(p_{1,1}+p_{1,2})p_{2,2}p_{4,2}p_{5,1}.$$

and the reliability of LMCCS-MN is

$$R=1-F=1-p_{2,1}-p_{2,2}p_{4,1}-(p_{1,1}+p_{1,2})p_{2,2}p_{4,2}p_{5,1}.$$

5.2. Numerical example

Considering a linear wireless sensor network, sensors are allocated to the non-uniform distributed nodes along a line. The linear wireless sensor network consists of $N+1=11$ nodes. The distances S_j between node j and the first node are shown in Table 1. Each of the first 10 nodes can contain a sensor, which provides a downstream connection with other nodes along a line. The linear wireless sensor network can be seen as a simple LMCCS-MN model. It fails if there exist at least m consecutive gaps or n total gaps. The PMF of X_j and the corresponding G_j for all the sensors are listed in Table 2. The nodes 3 and 6 are empty, which is actually equivalent to allocating dummy sensors represented with $\Pr\{X_j=0\}=1$.

Table 3 gives the values of the system reliability of LMCCS-MN as function m and n. The values of the system reliability in the first row and the first column are all equivalent to 0.02514. This indicates that the LMCCS-MN reduces to the traditional LMCCS model when $m=1$ or $n=1$, where no gap is allowed. The value in the third row and the fourth column is 0.5623, which means the reliability of the LMCCS-MN system with the constraints of $m=3$ consecutive gaps and $n=4$ total gaps in the example. In the last column the constraint of $n=N$ total gaps is in vain and only the constraint of m consecutive gaps is effective. The LMCCS-MN system model with $n=N$ is still different from the mGCCS model [23]. The proposed LMCCS-MN model can deal with the non-uniform deployment of the system nodes, while the mGCCS model only deals with the uniform deployment of system nodes. The mGCCS model is a particular case of the proposed LMCCS-MN model. In each column, all the entries below the main diagonal are identical to the one on the main diagonal. In fact, the number of consecutive gaps is no more than the number of total gaps in the LMCCS-MN system. Thus, when $m\geq n$, the LMCCS-MN model is only subject to the constraint of n total gaps. A particular case of this situation is the LMCCS-N model [2]. The LMCCS-N model only deals with the uniform deployment of system nodes. The entries above the main diagonal are of great interest. This shows that when $1<m<n<N$, the LMCCS-MN model is subject to the dual constraints. A particular case of this situation is the m/nCCS model [24], which is subject to the combined gap constraints. The m/nCCS model allows uniform deployment of system nodes, while the proposed LMCCS-MN model allows non-uniform deployment of system nodes. Fig. 4 indicates that the reliability of the model first increases with the constraint of m when $m<n$ and eventually reaches an a stable value when $m\geq n$. Fig. 5 shows that the reliability of the model first increases with the constraint of n and eventually reaches a stable value.

Table 1
The distance S_j between node j and the first node.

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}
Distances	0	4	6	10	18	26	30	36	40	44	50

Table 2
PMF of X_j and G_j for the numerical example.

Index of sensors	State no.	1	2	3	4	5
1	P	0.3	0.7	–	–	–
	X	0	5	–	–	–
	G	0	5	–	–	–
2	P	0.2	0.1	0.4	0.3	–
	X	0	2	4	6	–
	G	4	6	8	10	–
3	P	1.0	–	–	–	–
	X	0	–	–	–	–
	G	6	–	–	–	–
4	P	0.25	0.05	0.4	0.3	–
	X	0	4	8	10	–
	G	10	14	18	20	–
5	P	0.08	0.2	0.15	0.45	0.12
	X	0	4	8	12	16
	G	18	22	26	30	34
6	P	1.0	–	–	–	–
	X	0	–	–	–	–
	G	26	–	–	–	–
7	P	0.3	0.1	0.1	0.5	–
	X	0	4	5	7	–
	G	30	34	35	37	–
8	P	0.05	0.25	0.7	–	–
	X	0	5	10	–	–
	G	36	41	46	–	–
9	P	0.6	0.4	–	–	–
	X	0	6	–	–	–
	G	40	46	–	–	–
10	P	0.25	0.75	–	–	–
	X	0	8	–	–	–
	G	44	50	–	–	–

5.3. Optimal element allocation of the numerical example

Subject to the given sensors, the reliability of the system can be greatly affected by the allocation of sensors among the system nodes. The reliability of the system can be enhanced by reallocating the sensors to the system nodes. The optimal allocation for the LMCCS-MN is very valuable and can provide some good reference for the design of LMCCS-MN systems. The optimal allocation problem of N sensors in the LMCCS-MN with N nodes is a complicated optimization problem. The number of all possible solutions for the problem is $N!$. Considering the algorithm of reliability evaluation, the computational complexity of the optimal allocation problem is $O\left(mnN^3\times N!\right)$. It is not practical to use the brute-force with an exhaustive examination of all possible solutions in a reasonable time constraint. The heuristic search algorithms are widely applied to optimize engineering problems. As a heuristic search algorithm, genetic algorithm (GA) has strong global search ability, and has been proved an efficient optimization tool in reliability engineering [22]. In this paper, we adopt the package for GA in Ref. [33]. Without optimal allocation, the reliability of the system in the Table 3 in the Section 5.2 is 0.56230, when $m=3$ and $n=4$. Table 4 gives two optimal element

Table 3
LMCCS-MN reliability as function of m and n.

m	n									
	1	2	3	4	5	6	7	8	9	10
1	**0.02514**	0.02514	0.02514	0.02514	0.02514	0.02514	0.02514	0.02514	0.02514	0.02514
2	0.02514	**0.13405**	0.2883	0.37187	0.3872	0.38725	0.38725	0.38725	0.38725	0.38725
3	0.02514	0.13405	**0.34512**	0.5623	0.68492	0.71978	0.72439	0.72452	0.72452	0.72452
4	0.02514	0.13405	0.34512	**0.59573**	0.78552	0.876	0.90216	0.90617	0.90643	0.90643
5	0.02514	0.13405	0.34512	0.59573	**0.80358**	0.91542	0.95129	0.95759	0.95807	0.95807
6	0.02514	0.13405	0.34512	0.59573	0.80358	**0.92844**	0.9759	0.98594	0.98691	0.98692
7	0.02514	0.13405	0.34512	0.59573	0.80358	0.92844	**0.98151**	0.9948	0.99639	0.99646
8	0.02514	0.13405	0.34512	0.59573	0.80358	0.92844	0.98151	**0.99679**	0.99937	0.99954
9	0.02514	0.13405	0.34512	0.59573	0.80358	0.92844	0.98151	0.99679	**0.99965**	0.99988
10	0.02514	0.13405	0.34512	0.59573	0.80358	0.92844	0.98151	0.99679	0.99965	**1**

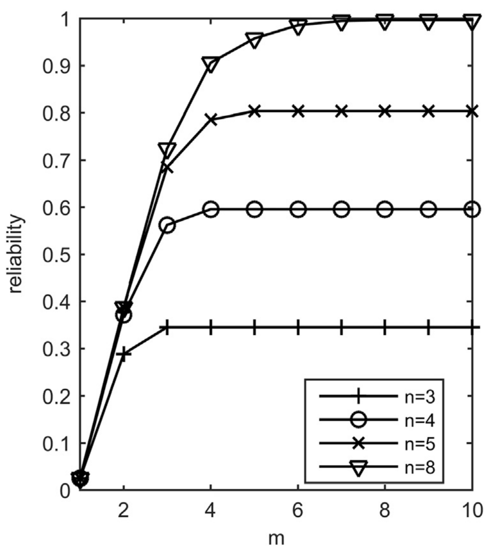

Fig. 4. Reliability as function of ***m*** for different ***n***.

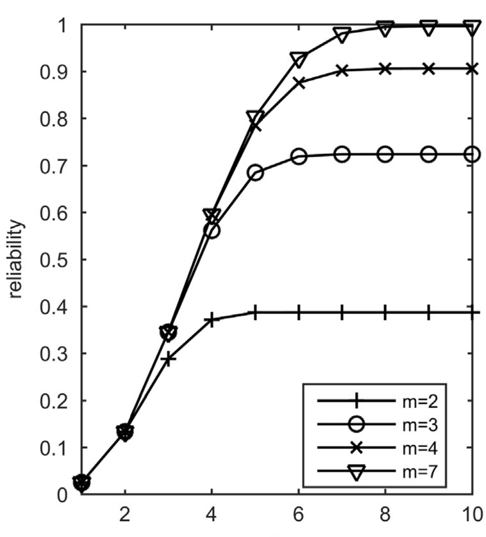

Fig. 5. Reliability as function of ***n*** for different ***m***.

allocation of the numerical example, when $m = 3$ and $n = 4$. After applying the optimal element allocation, the reliability of the LMCCS-MN in the numerical example in the Section 5.2 is raised to 0.77888. The reliability of the LMCCS-MN increases 38.52% by the optimal element allocation. Meanwhile, it can be seen that the optimal allocation can be more than one. The reason is that two of the nodes in the numerical example are empty. They can be seen as two dummy sensors, which remain the failure state with probability being equal to 1. The sequences of dummy sensors do not affect the reliability of system. Thus, different kinds of allocations can result in the same optimal reliability.

Table 4
Optimal allocation of the numerical example, when $m = 3$ and $n = 4$.

Allocations	**Nodes index**										**Reliability**
	1	2	3	4	5	6	7	8	9	10	
1	7	2	5	6	10	8	3	4	1	9	0.77888
2	7	2	5	3	10	8	6	4	1	9	0.77888

6. Conclusions

Considering the dual constraints of m consecutive gaps and n total gaps in practice, the paper introduces the LMCCS-MN model, which is a generalized model of the LMCCS. The proposed LMCCS-MN model deals with the non-uniform deployment of system nodes. The LMCCS-MN fails if it meets either of the two constraints. In this paper, two different types of failures: the failure of consecutive gaps and the failure of total gaps are studied. In order to evaluate the system reliability, the universal generating function technique is adopted in the reliability evaluation algorithm. The computational complexity of the suggested algorithm is polynomial and is less than $O\left(mnN^3\right)$, where m is the allowed consecutive number of gaps, n is the allowed total number of gaps, and N is the number of nodes in the system. The algorithm can be used to evaluate the system reliability and to search the optimal allocations of the LMCCS-MN, which is shown by illustrative examples in Section 5.

The proposed LMCCS-MN model can be applied to the fluid transportation systems, wireless communication systems, sensor systems and logistics systems. One limitation of the proposed LMCCS-MN model is that it is only effective and suitable for the linear consecutively-connected system with one source and one sink. In real engineering applications, some wireless sensor networks would provide connection between one source node and more than one sink nodes or between one sink node and more than one source nodes [34–36]. In the further researches, the LMCCS-MN model can be extended to evaluate the wireless sensor networks with multiple source nodes or multiple sink nodes, and the circular multi-state consecutively-connected system. If a gap exists before the node j in the LMCC-MN model, the distance between the location of the node j and the most remote location covered by the nodes ahead of the node j is referred to as the size of the gap. In the proposed LMCCS-MN model, we consider the unevenly distributed nodes subject to the number of gaps constraints, but we do not involve the size of a gap. In the further researches, the size of a gap constraint should be involved in the LMCCS model. In addition, the phased-mission and elements maintenance can be taken into account in the future searches.

Acknowledgment

This work is supported by the NSFC under grant number 71231001, 71301009 and 71420107023, and by the Fundamental Research Funds for the Central Universities of China Grant YWF-14-KKX-008, FRF-TP-15-031A3 and FRF-BR-15-001B.

Appendix

According to the algorithm of reliability evaluation in Section 3 and the u-functions of the nodes, assign $U_0(z)=z^{(0,0,0)}$,$F=0$, and apply the operator $\otimes_\Omega$ in (11) recursively. $U_1(z), U_2(z), U_3(z), U_4(z)$, $U_5(z)$ and truncated u-functions $\tilde{U}_2(z), \tilde{U}_3(z), \tilde{U}_4(z)$ are obtained as follows:

For $j=1$: $S_2=2$

$$U_1(z)=U_0(z)\otimes_\Omega u_1(z)=z^{(0,0,0)}\otimes_\Omega(p_{1,1}z^0+p_{1,2}z^1+p_{1,3}z^2)$$
$$=p_{1,1}z^{(0,1,1)}+p_{1,2}z^{(1,1,1)}+p_{1,3}z^{(2,0,0)}.$$

For $j=2$: $S_3=3$

$$\begin{aligned}U_2(z)&=U_1(z)\otimes_\Omega u_2(z)\\&=(p_{1,1}z^{(0,1,1)}+p_{1,2}z^{(1,1,1)}+p_{1,3}z^{(2,0,0)})\otimes_\Omega(p_{2,1}z^2+p_{2,2}z^3)\\&=p_{1,1}p_{2,1}z^{(2,2,2)}+p_{1,2}p_{2,1}z^{(2,2,2)}+p_{1,3}p_{2,1}z^{(2,1,1)}+p_{1,1}p_{2,2}z^{(3,0,1)}\\&\quad+p_{1,2}p_{2,2}z^{(3,0,1)}+p_{1,3}p_{2,2}z^{(3,0,0)}=(p_{1,1}+p_{1,2})p_{2,1}z^{(2,2,2)}\\&\quad+p_{1,3}p_{2,1}z^{(2,1,1)}+(p_{1,1}+p_{1,2})p_{2,2}z^{(3,0,1)}+p_{1,3}p_{2,2}z^{(3,0,0)}.\end{aligned}$$

The term $(p_{1,1}+p_{1,2})p_{2,1}z^{(2,2,2)}$ corresponds to the failure of LMCCS-MN. The value $\psi_{2,3}(U_2(z))=(p_{1,1}+p_{1,2})p_{2,1}$ is added to F. After removing the term corresponding to the system failure, the truncated u-function $\tilde{U}_2(z)$ is

$$\tilde{U}_2(z)=p_{1,3}p_{2,1}z^{(2,1,1)}+(p_{1,1}+p_{1,2})p_{2,2}z^{(3,0,1)}+p_{1,3}p_{2,2}z^{(3,0,0)}.$$

For $j=3$: $S_4=4$

$$\begin{aligned}U_3(z)&=\tilde{U}_2(z)\otimes_\Omega u_3(z)=(p_{1,3}p_{2,1}z^{(2,1,1)}+(p_{1,1}+p_{1,2})p_{2,2}z^{(3,0,1)}\\&\quad+p_{1,3}p_{2,2}z^{(3,0,0)})\otimes_\Omega(z^3)=p_{1,3}p_{2,1}z^{(3,2,2)}+(p_{1,1}+p_{1,2})p_{2,2}z^{(3,1,2)}\\&\quad+p_{1,3}p_{2,2}z^{(3,1,1)}.\end{aligned}$$

The term $p_{1,3}p_{2,1}z^{(3,2,2)}$ corresponds to the failure of LMCCS-MN. The value $\psi_{2,3}(U_3(z))=p_{1,3}p_{2,1}$ is added to F. After removing the term corresponding to the system failure, the truncated u-function $\tilde{U}_3(z)$ is

$$\tilde{U}_3(z)=(p_{1,1}+p_{1,2})p_{2,2}z^{(3,1,2)}+p_{1,3}p_{2,2}z^{(3,1,1)}.$$

For $j=4$: $S_5=5$

$$\begin{aligned}U_4(z)&=\tilde{U}_3(z)\otimes_\Omega u_4(z)\\&=((p_{1,1}+p_{1,2})p_{2,2}z^{(3,1,2)}+p_{1,3}p_{2,2}z^{(3,1,1)})\otimes_\Omega(p_{4,1}z^4+p_{4,2}z^5)\\&=(p_{1,1}+p_{1,2})p_{2,2}p_{4,1}z^{(4,2,3,)}+p_{1,3}p_{2,2}p_{4,1}z^{(4,2,2)}\\&\quad+(p_{1,1}+p_{1,2})p_{2,2}p_{4,2}z^{(5,0,2)}+p_{1,3}p_{2,2}p_{4,2}z^{(5,0,1)}.\end{aligned}$$

The term $(p_{1,1}+p_{1,2})p_{2,2}p_{4,1}z^{(4,2,3,)}$ and term $p_{1,3}p_{2,2}p_{4,1}z^{(4,2,2)}$ correspond to the failure of LMCCS-MN. The value $\psi_{2,3}(U_4(z))=(p_{1,1}+p_{1,2})p_{2,2}p_{4,1}+p_{1,3}p_{2,2}p_{4,1}=p_{2,2}p_{4,1}$ is added to F. After removing the terms corresponding to the system failure, the truncated u-function $\tilde{U}_4(z)$ is

$$\tilde{U}_4(z)=(p_{1,1}+p_{1,2})p_{2,2}p_{4,2}z^{(5,0,2)}+p_{1,3}p_{2,2}p_{4,2}z^{(5,0,1)}.$$

For $j=5$: $S_6=6$

$$\begin{aligned}U_5(z)&=\tilde{U}_4(z)\otimes_\Omega u_5(z)\\&=((p_{1,1}+p_{1,2})p_{2,2}p_{4,2}z^{(5,0,2)}+p_{1,3}p_{2,2}p_{4,2}z^{(5,0,1)})\otimes_\Omega(p_{5,1}z^5+p_{5,2}z^6)\\&=(p_{1,1}+p_{1,2})p_{2,2}p_{4,2}p_{5,1}z^{(5,1,3)}+p_{1,3}p_{2,2}p_{4,2}p_{5,1}z^{(5,1,2)}\\&\quad+(p_{1,1}+p_{1,2})p_{2,2}p_{4,2}p_{5,2}z^{(6,0,2)}+p_{1,3}p_{2,2}p_{4,2}p_{5,2}z^{(6,0,1)}.\end{aligned}$$

The term $(p_{1,1}+p_{1,2})p_{2,2}p_{4,2}p_{5,1}z^{(5,1,3)}$ corresponds to the failure of LMCCS-MN. The value $\psi_{2,3}(U_5(z))=(p_{1,1}+p_{1,2})p_{2,2}p_{4,2}p_{5,1}$ is added to F. Hence, the failure probability of LMCCS-MN is

$$F=p_{2,1}+p_{2,2}p_{4,1}+(p_{1,1}+p_{1,2})p_{2,2}p_{4,2}p_{5,1}.$$

References

[1] Hwang FK, Yao YC. Multistate consecutively-connected systems. IEEE Trans Reliab 1989;38:472–4. http://dx.doi.org/10.1109/24.46467.
[2] Levitin G, Xing L, Dai Y. Linear multistate consecutively-connected systems subject to a constrained number of gaps. Reliab Eng Syst Saf 2015;133:246–52. http://dx.doi.org/10.1016/j.ress.2014.09.004.
[3] Zuo MJ. Reliability of linear and circular consecutively-connected systems. IEEE Trans Reliab 1993;42:484–7. http://dx.doi.org/10.1109/24.257835.
[4] Kossow A, Preuss W. Reliability of linear consecutively-connected systems with multistate components. IEEE Trans Reliab 1995;44:518–22. http://dx.doi.org/10.1109/24.406595.
[5] Yeh W-C. Multistate-node acyclic network reliability evaluation. Reliab Eng Syst Saf 2002;78:123–9. http://dx.doi.org/10.1016/S0951-8320(02)00114-X.
[6] Levitin G. Reliability evaluation for linear consecutively-connected systems with multistate elements and retransmission delays. Qual Reliab Eng Int 2001;17:373–8. http://dx.doi.org/10.1002/qre.418.
[7] Levitin G, Yeh W-C, Dai Y. Minimizing bypass transportation expenses in linear multistate consecutively-connected systems. IEEE Trans Reliab 2014;63:230–8. http://dx.doi.org/10.1109/TR.2014.2299156.
[8] Pootakham T, Kumar A. A comparison of pipeline versus truck transport of bio-oil. Bioresour Technol 2010;101:414–21. http://dx.doi.org/10.1016/j.biortech.2009.07.077.
[9] Pootakham T, Kumar A. Bio-oil transport by pipeline: a techno-economic assessment. Bioresour Technol 2010;101:7137–43. http://dx.doi.org/10.1016/j.biortech.2010.03.136.
[10] Zio E, Marella M, Podofillini L. Importance measures-based prioritization for improving the performance of multi-state systems: application to the railway industry. Reliab Eng Syst Saf 2007;92:1303–14. http://dx.doi.org/10.1016/j.ress.2006.07.010.
[11] Soszynska J. Reliability and risk evaluation of a port oil pipeline transportation system in variable operation conditions. Int J Press Vessels Pip 2010;87:81–7. http://dx.doi.org/10.1016/j.ijpvp.2010.01.002.
[12] Soszynska J. Reliability evaluation of a port oil transportation system in variable operation conditions. Int J Press Vessels Pip 2006;83:304–11. http://dx.doi.org/10.1016/j.ijpvp.2006.02.011.
[13] Macchi M, Garetti M, Centrone D, Fumagalli L, Piero Pavirani G. Maintenance management of railway infrastructures based on reliability analysis. Reliab Eng Syst Saf 2012;104:71–83. http://dx.doi.org/10.1016/j.ress.2012.03.017.
[14] Malinowski J, Preuss W. Reliability increase of consecutive k-out-of-n:f and related systems through components' rearrangement. Microelectron Reliab 1996;36:1417–23. http://dx.doi.org/10.1016/0026-2714(96)00004-2.
[15] Levitin G. Optimal allocation of multistate elements in a linear consecutively-connected system. IEEE Trans Reliab 2003;52:192–9. http://dx.doi.org/10.1109/TR.2003.809655.
[16] Levitin G. Optimal allocation of multi-state elements in linear consecutively connected systems with vulnerable nodes. Eur J Oper Res 2003;150:406–19. http://dx.doi.org/10.1016/S0377-2217(02)00506-4.
[17] Peng R, Xie M, Ng SH, Levitin G. Element maintenance and allocation for linear consecutively connected systems. IIE Trans 2012;44:964–73. http://dx.doi.org/10.1080/0740817X.2011.649388.
[18] Levitin G, Xing L, Dai Y. Optimal allocation of connecting elements in phase mission linear consecutively-connected systems. IEEE Trans Reliab 2013;62:618–27. http://dx.doi.org/10.1109/TR.2013.2270413.
[19] Peng R, Zhai Q, Xing L, Yang J. Reliability of demand-based phased-mission systems subject to fault level coverage. Reliab Eng Syst Saf 2014;121:18–25. http://dx.doi.org/10.1016/j.ress.2013.07.013.
[20] Mo Y, Xing L, Amari SV. A multiple-valued decision diagram based method for efficient reliability analysis of non-repairable phased-mission systems. IEEE Trans Reliab 2014;63:320–30. http://dx.doi.org/10.1109/TR.2014.2299497.
[21] Xing L, Amari SV, Wang C. Reliability of k-out-of-n systems with phased-mission requirements and imperfect fault coverage. Reliab Eng Syst Saf 2012;103:45–50. http://dx.doi.org/10.1016/j.ress.2012.03.018.

[22] Levitin G, Xing L, Yu S. Optimal connecting elements allocation in linear consecutively-connected systems with phased mission and common cause failures. Reliab Eng Syst Saf 2014;130:85–94. http://dx.doi.org/10.1016/j.ress.2014.04.028.
[23] Xiang Y, Levitin G, Dai Y. Linear multistate consecutively-connected systems with gap constraints. IEEE Trans Reliab 2012;61:208–14. http://dx.doi.org/10.1109/TR.2011.2182393.
[24] Levitin G, Xing L, Ben-Haim H, Dai Y. m/nCCS: linear consecutively connected systems subject to combined gap constraints. Int J Gen Syst 2015;0:1–16. http://dx.doi.org/10.1080/03081079.2015.1011633.
[25] Ramirez-Marquez JE, Levitin G. Algorithm for estimating reliability confidence bounds of multi-state systems. Reliab Eng Syst Saf 2008;93:1231–43. http://dx.doi.org/10.1016/j.ress.2007.07.003.
[26] Ding Y, Lisnianski A. Fuzzy universal generating functions for multi-state system reliability assessment. Fuzzy Sets Syst 2008;159:307–24. http://dx.doi.org/10.1016/j.fss.2007.06.004.
[27] Ding Y, Zuo MJ, Tian Z, Li W. The hierarchical weighted multi-state k-out-of-n system model and its application for infrastructure management. IEEE Trans Reliab 2010;59:593–603. http://dx.doi.org/10.1109/TR.2010.2054171.
[28] Destercke S, Sallak M. An extension of universal generating function in multi-state systems considering epistemic uncertainties. IEEE Trans Reliab 2013;62:504–14. http://dx.doi.org/10.1109/TR.2013.2259206.
[29] Li YF, Peng R. Availability modeling and optimization of dynamic multi-state series–parallel systems with random reconfiguration. Reliab Eng Syst Saf 2014;127:47–57. http://dx.doi.org/10.1016/j.ress.2014.03.005.
[30] Yu H, Yang J, Mo H. Reliability analysis of repairable multi-state system with common bus performance sharing. Reliab Eng Syst Saf 2014;132:90–6. http://dx.doi.org/10.1016/j.ress.2014.07.017.
[31] Yeh W-C. A modified universal generating function algorithm for the acyclic binary-state network reliability. IEEE Trans Reliab 2012;61:702–9. http://dx.doi.org/10.1109/TR.2012.2207574.
[32] Levitin G. The universal generating function in reliability analysis and optimization. London: Springer; 2005.
[33] Scrucca LG. A Package for Genetic Algorithms in R. J Stat Softw 2013;53:1–37.
[34] Halder S, Ghosal A, Bit SD. A pre-determined node deployment strategy to prolong network lifetime in wireless sensor network. Comput Commun 2011;34:1294–306. http://dx.doi.org/10.1016/j.comcom.2011.01.004.
[35] Halder S, Das Bit S. Enhancement of wireless sensor network lifetime by deploying heterogeneous nodes. J Netw Comput Appl 2014;38:106–24. http://dx.doi.org/10.1016/j.jnca.2013.03.008.
[36] Angelopoulos CM, Nikoletseas S, Raptis TP. Wireless energy transfer in sensor networks with adaptive, limited knowledge protocols. Comput Netw 2014;70:113–41. http://dx.doi.org/10.1016/j.comnet.2014.04.022.