In calculus it is sometimes necessary to treat a function as a composition of two functions. The next example shows how this can be done.

EXAMPLE 9 Finding Functions That Form a Given Composite

Find functions f and g such that

$$(f \circ g)(x) = (x^2 - 5)^3 - 4(x^2 - 5) + 3.$$

SOLUTION Note the repeated quantity $x^2 - 5$. If we choose $g(x) = x^2 - 5$ and $f(x) = x^3 - 4x + 3$, then we have the following.

$$\begin{aligned}
&(f \circ g)(x) \\
&= f(g(x)) && \text{By definition} \\
&= f(x^2 - 5) && g(x) = x^2 - 5 \\
&= (x^2 - 5)^3 - 4(x^2 - 5) + 3 && \text{Use the rule for } f.
\end{aligned}$$

There are other pairs of functions f and g that also satisfy these conditions. For instance,

$$f(x) = (x - 5)^3 - 4(x - 5) + 3 \quad \text{and} \quad g(x) = x^2.$$

✔ **Now Try Exercise 99.**

2.8 Exercises

CONCEPT PREVIEW *Without using paper and pencil, evaluate each expression given the following functions.*

$$f(x) = x + 1 \quad \text{and} \quad g(x) = x^2$$

1. $(f + g)(2)$ **2.** $(f - g)(2)$ **3.** $(fg)(2)$

4. $\left(\frac{f}{g}\right)(2)$ **5.** $(f \circ g)(2)$ **6.** $(g \circ f)(2)$

CONCEPT PREVIEW *Refer to functions f and g as described in **Exercises 1–6**, and find the following.*

7. domain of f **8.** domain of g **9.** domain of $f + g$ **10.** domain of $\frac{f}{g}$

*Let $f(x) = x^2 + 3$ and $g(x) = -2x + 6$. Find each of the following. **See Example 1.***

11. $(f + g)(3)$ **12.** $(f + g)(-5)$ **13.** $(f - g)(-1)$ **14.** $(f - g)(4)$

15. $(fg)(4)$ **16.** $(fg)(-3)$ **17.** $\left(\frac{f}{g}\right)(-1)$ **18.** $\left(\frac{f}{g}\right)(5)$

*For the pair of functions defined, find $(f + g)(x)$, $(f - g)(x)$, $(fg)(x)$, and $\left(\frac{f}{g}\right)(x)$. Give the domain of each. **See Example 2.***

19. $f(x) = 3x + 4,\ g(x) = 2x - 5$ **20.** $f(x) = 6 - 3x,\ g(x) = -4x + 1$

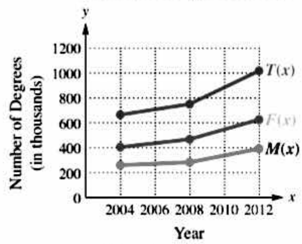
21. $f(x) = 2x^2 - 3x,\ g(x) = x^2 - x + 3$ **22.** $f(x) = 4x^2 + 2x,\ g(x) = x^2 - 3x + 2$

23. $f(x) = \sqrt{4x - 1},\ g(x) = \frac{1}{x}$ **24.** $f(x) = \sqrt{5x - 4},\ g(x) = -\frac{1}{x}$

Associate's Degrees Earned The graph shows the number of associate's degrees earned (in thousands) in the United States from 2004 through 2012.

$M(x)$ gives the number of degrees earned by males.
$F(x)$ gives the number earned by females.
$T(x)$ gives the total number for both groups.

Associate's Degrees Earned



Source: National Center for Education Statistics.

25. Estimate $M(2008)$ and $F(2008)$, and use the results to estimate $T(2008)$.

26. Estimate $M(2012)$ and $F(2012)$, and use the results to estimate $T(2012)$.

27. Use the slopes of the line segments to decide in which period (2004–2008 or 2008–2012) the total number of associate's degrees earned increased more rapidly.

28. *Concept Check* Refer to the graph of Associate's Degrees Earned.

If $2004 \leq k \leq 2012$, $T(k) = r$, and $F(k) = s$, then $M(k) =$ ______.

Science and Space/Technology Spending The graph shows dollars (in billions) spent for general science and for space/other technologies in selected years.

$G(x)$ represents the dollars spent for general science.
$S(x)$ represents the dollars spent for space and other technologies.
$T(x)$ represents the total expenditures for these two categories.

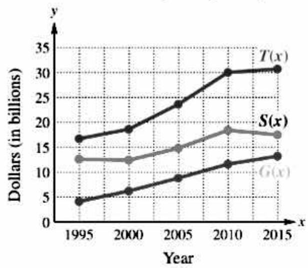
29. Estimate $(T - S)(2000)$. What does this function represent?

30. Estimate $(T - G)(2010)$. What does this function represent?

31. In which category and which period(s) does spending decrease?

32. In which period does spending for $T(x)$ increase most?

Science and Space Spending

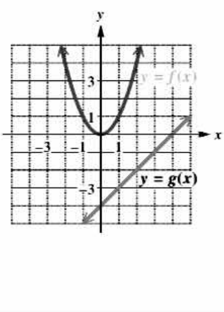


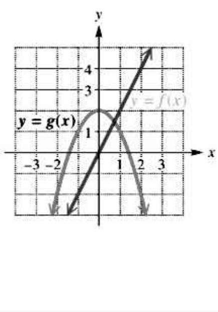
Source: U.S. Office of Management and Budget.

Use the graph to evaluate each expression. See ***Example 3(a).***

33. (a) $(f + g)(2)$ **(b)** $(f - g)(1)$ **(c)** $(fg)(0)$ **(d)** $\left(\frac{f}{g}\right)(1)$

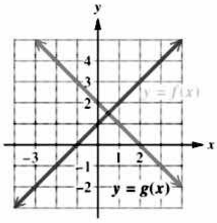
34. (a) $(f + g)(0)$ **(b)** $(f - g)(-1)$ **(c)** $(fg)(1)$ **(d)** $\left(\frac{f}{g}\right)(2)$





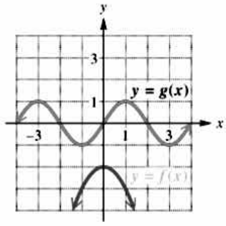
35. **(a)** $(f+g)(-1)$ **(b)** $(f-g)(-2)$

(c) $(fg)(0)$ **(d)** $\left(\frac{f}{g}\right)(2)$



36. **(a)** $(f+g)(1)$ **(b)** $(f-g)(0)$

(c) $(fg)(-1)$ **(d)** $\left(\frac{f}{g}\right)(1)$



Use the table to evaluate each expression in parts (a)–(d), if possible. ***See Example 3(b).***

(a) $(f+g)(2)$ **(b)** $(f-g)(4)$ **(c)** $(fg)(-2)$ **(d)** $\left(\frac{f}{g}\right)(0)$

37.

x	$f(x)$	$g(x)$
-2	0	6
0	5	0
2	7	-2
4	10	5

38.

x	$f(x)$	$g(x)$
-2	-4	2
0	8	-1
2	5	4
4	0	0

39. Use the table in **Exercise 37** to complete the following table.

x	$(f+g)(x)$	$(f-g)(x)$	$(fg)(x)$	$\left(\frac{f}{g}\right)(x)$
-2				
0				
2				
4				

40. Use the table in **Exercise 38** to complete the following table.

x	$(f+g)(x)$	$(f-g)(x)$	$(fg)(x)$	$\left(\frac{f}{g}\right)(x)$
-2				
0				
2				
4				

41. *Concept Check* How is the difference quotient related to slope?

42. *Concept Check* Refer to **Figure 93.** How is the secant line PQ related to the tangent line to a curve at point P?

For each function, find ***(a)*** $f(x+h)$, ***(b)*** $f(x+h)-f(x)$, *and* ***(c)*** $\dfrac{f(x+h)-f(x)}{h}$.

See Example 4.

43. $f(x)=2-x$

44. $f(x)=1-x$

45. $f(x)=6x+2$

46. $f(x)=4x+11$

47. $f(x)=-2x+5$

48. $f(x)=-4x+2$

49. $f(x) = \frac{1}{x}$ 50. $f(x) = \frac{1}{x^2}$ 51. $f(x) = x^2$

52. $f(x) = -x^2$ 53. $f(x) = 1 - x^2$ 54. $f(x) = 1 + 2x^2$

55. $f(x) = x^2 + 3x + 1$ 56. $f(x) = x^2 - 4x + 2$

Let $f(x) = 2x - 3$ *and* $g(x) = -x + 3$. *Find each function value.* ***See Example 5.***

57. $(f \circ g)(4)$ 58. $(f \circ g)(2)$ 59. $(f \circ g)(-2)$ 60. $(g \circ f)(3)$

61. $(g \circ f)(0)$ 62. $(g \circ f)(-2)$ 63. $(f \circ f)(2)$ 64. $(g \circ g)(-2)$

Concept Check The tables give some selected ordered pairs for functions f and g.

x	3	4	6
$f(x)$	1	3	9

x	2	7	1	9
$g(x)$	3	6	9	12

Find each of the following.

65. $(f \circ g)(2)$ 66. $(f \circ g)(7)$ 67. $(g \circ f)(3)$

68. $(g \circ f)(6)$ 69. $(f \circ f)(4)$ 70. $(g \circ g)(1)$

71. *Concept Check* Why can we not determine $(f \circ g)(1)$ given the information in the tables for **Exercises 65–70?**

72. *Concept Check* Extend the concept of composition of functions to evaluate $(g \circ (f \circ g))(7)$ using the tables for **Exercises 65–70.**

Given functions f and g, find ***(a)*** $(f \circ g)(x)$ *and its domain, and* ***(b)*** $(g \circ f)(x)$ *and its domain.* ***See Examples 6 and 7.***

73. $f(x) = -6x + 9,\ g(x) = 5x + 7$ 74. $f(x) = 8x + 12,\ g(x) = 3x - 1$

75. $f(x) = \sqrt{x},\ g(x) = x + 3$ 76. $f(x) = \sqrt{x},\ g(x) = x - 1$

77. $f(x) = x^3,\ g(x) = x^2 + 3x - 1$ 78. $f(x) = x + 2,\ g(x) = x^4 + x^2 - 4$

79. $f(x) = \sqrt{x - 1},\ g(x) = 3x$ 80. $f(x) = \sqrt{x - 2},\ g(x) = 2x$

81. $f(x) = \frac{2}{x},\ g(x) = x + 1$ 82. $f(x) = \frac{4}{x},\ g(x) = x + 4$

83. $f(x) = \sqrt{x + 2},\ g(x) = -\frac{1}{x}$ 84. $f(x) = \sqrt{x + 4},\ g(x) = -\frac{2}{x}$

85. $f(x) = \sqrt{x},\ g(x) = \frac{1}{x + 5}$ 86. $f(x) = \sqrt{x},\ g(x) = \frac{3}{x + 6}$

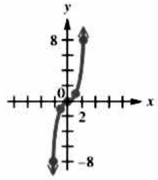
87. $f(x) = \frac{1}{x - 2},\ g(x) = \frac{1}{x}$ 88. $f(x) = \frac{1}{x + 4},\ g(x) = -\frac{1}{x}$

89. *Concept Check* Fill in the missing entries in the table.

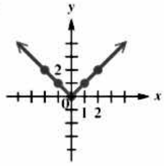
x	$f(x)$	$g(x)$	$g(f(x))$
1	3	2	7
2	1	5	
3	2		

4. For each basic function graphed, give the name of the function, the domain, the range, and open intervals over which it is decreasing, increasing, or constant.

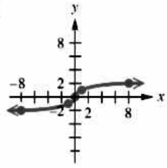
(a)



(b)



(c)



5. *(Modeling) Long-Distance Call Charges* A certain long-distance carrier provides service between Podunk and Nowheresville. If x represents the number of minutes for the call, where $x > 0$, then the function

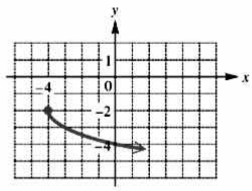
$$f(x) = 0.40[\![x]\!] + 0.75$$

gives the total cost of the call in dollars. Find the cost of a 5.5-min call.

Graph each function.

6. $f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ 2x + 3 & \text{if } x < 0 \end{cases}$ 7. $f(x) = -x^3 + 1$ 8. $f(x) = 2|x - 1| + 3$

9. *Connecting Graphs with Equations* The function $g(x)$ graphed here is obtained by stretching, shrinking, reflecting, and/or translating the graph of $f(x) = \sqrt{x}$. Give the equation that defines this function.



10. Determine whether each function is *even*, *odd*, or *neither*.

(a) $f(x) = x^2 - 7$ (b) $f(x) = x^3 - x - 1$ (c) $f(x) = x^{101} - x^{99}$

2.8 Function Operations and Composition

- **Arithmetic Operations on Functions**
- **The Difference Quotient**
- **Composition of Functions and Domain**

DVD Production

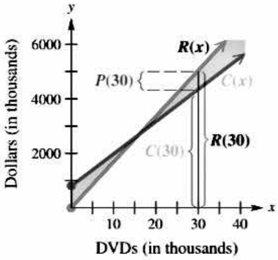


Figure 92

Arithmetic Operations on Functions

Figure 92 shows the situation for a company that manufactures DVDs. The two lines are the graphs of the linear functions for

$$\text{revenue } R(x) = 168x \quad \text{and} \quad \text{cost } C(x) = 118x + 800,$$

where x is the number of DVDs produced and sold, and x, $R(x)$, and $C(x)$ are given in thousands. When 30,000 (that is, 30 thousand) DVDs are produced and sold, profit is found as follows.

$$P(x) = R(x) - C(x) \qquad \text{Profit function}$$
$$P(30) = R(30) - C(30) \qquad \text{Let } x = 30.$$
$$P(30) = 5040 - 4340 \qquad R(30) = 168(30);\ C(30) = 118(30) + 800$$
$$P(30) = 700 \qquad \text{Subtract.}$$

Thus, the profit from the sale of 30,000 DVDs is \$700,000.

The profit function is found by *subtracting* the cost function from the revenue function. New functions can be formed by using other operations as well.

Operations on Functions and Domains

Given two functions f and g, then for all values of x for which both $f(x)$ and $g(x)$ are defined, the functions $f + g$, $f - g$, fg, and $\frac{f}{g}$ are defined as follows.

$$(f + g)(x) = f(x) + g(x) \quad \text{Sum function}$$

$$(f - g)(x) = f(x) - g(x) \quad \text{Difference function}$$

$$(fg)(x) = f(x) \cdot g(x) \quad \text{Product function}$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0 \quad \text{Quotient function}$$

The **domains of $f + g$, $f - g$,** and **fg** include all real numbers in the intersection of the domains of f and g, while the **domain of $\frac{f}{g}$** includes those real numbers in the intersection of the domains of f and g for which $g(x) \neq 0$.

NOTE The condition $g(x) \neq 0$ in the definition of the quotient means that the domain of $\left(\frac{f}{g}\right)(x)$ is restricted to all values of x for which $g(x)$ is not 0. The condition does *not* mean that $g(x)$ is a function that is never 0.

EXAMPLE 1 Using Operations on Functions

Let $f(x) = x^2 + 1$ and $g(x) = 3x + 5$. Find each of the following.

(a) $(f + g)(1)$ **(b)** $(f - g)(-3)$ **(c)** $(fg)(5)$ **(d)** $\left(\frac{f}{g}\right)(0)$

SOLUTION

(a) First determine $f(1) = 2$ and $g(1) = 8$. Then use the definition.

$$\begin{aligned} &(f + g)(1) \\ &= f(1) + g(1) && (f + g)(x) = f(x) + g(x) \\ &= 2 + 8 && f(1) = 1^2 + 1;\ g(1) = 3(1) + 5 \\ &= 10 && \text{Add.} \end{aligned}$$

(b) $(f - g)(-3)$

$$\begin{aligned} &= f(-3) - g(-3) && (f - g)(x) = f(x) - g(x) \\ &= 10 - (-4) && f(-3) = (-3)^2 + 1;\ g(-3) = 3(-3) + 5 \\ &= 14 && \text{Subtract.} \end{aligned}$$

(c) $(fg)(5)$

$$\begin{aligned} &= f(5) \cdot g(5) && (fg)(x) = f(x) \cdot g(x) \\ &= (5^2 + 1)(3 \cdot 5 + 5) && f(x) = x^2 + 1;\ g(x) = 3x + 5 \\ &= 26 \cdot 20 && f(5) = 26;\ g(5) = 20 \\ &= 520 && \text{Multiply.} \end{aligned}$$

(d) $\left(\frac{f}{g}\right)(0)$

$= \frac{f(0)}{g(0)}$ $\quad\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$= \frac{0^2 + 1}{3(0) + 5}$ $\quad f(x) = x^2 + 1$, $g(x) = 3x + 5$

$= \frac{1}{5}$ $\quad$ Simplify.

✔ **Now Try Exercises 11, 13, 15, and 17.**

EXAMPLE 2 Using Operations on Functions and Determining Domains

Let $f(x) = 8x - 9$ and $g(x) = \sqrt{2x - 1}$. Find each function in (a)–(d).

(a) $(f + g)(x)$ **(b)** $(f - g)(x)$ **(c)** $(fg)(x)$ **(d)** $\left(\frac{f}{g}\right)(x)$

(e) Give the domains of the functions in parts (a)–(d).

SOLUTION

(a) $(f + g)(x)$

$= f(x) + g(x)$

$= 8x - 9 + \sqrt{2x - 1}$

(b) $(f - g)(x)$

$= f(x) - g(x)$

$= 8x - 9 - \sqrt{2x - 1}$

(c) $(fg)(x)$

$= f(x) \cdot g(x)$

$= (8x - 9)\sqrt{2x - 1}$

(d) $\left(\frac{f}{g}\right)(x)$

$= \frac{f(x)}{g(x)}$

$= \frac{8x - 9}{\sqrt{2x - 1}}$

(e) To find the domains of the functions in parts (a)–(d), we first find the domains of f and g.

The domain of f is the set of all real numbers $(-\infty, \infty)$.

Because g is defined by a square root radical, the radicand must be non-negative (that is, greater than or equal to 0).

$g(x) = \sqrt{2x - 1}$ $\quad$ Rule for $g(x)$

$2x - 1 \geq 0$ $\quad$ $2x - 1$ must be nonnegative.

$x \geq \frac{1}{2}$ $\quad$ Add 1 and divide by 2.

Thus, the domain of g is $\left[\frac{1}{2}, \infty\right)$.

The domains of $f + g$, $f - g$, and fg are the intersection of the domains of f and g, which is

$$(-\infty, \infty) \cap \left[\frac{1}{2}, \infty\right) = \left[\frac{1}{2}, \infty\right).$$

The intersection of two sets is the set of all elements common to both sets.

The domain of $\frac{f}{g}$ includes those real numbers in the intersection above for which $g(x) = \sqrt{2x - 1} \neq 0$—that is, the domain of $\frac{f}{g}$ is $\left(\frac{1}{2}, \infty\right)$.

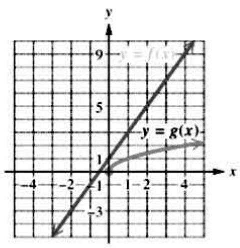
✔ **Now Try Exercises 19 and 23.**

EXAMPLE 3 Evaluating Combinations of Functions

If possible, use the given representations of functions f and g to evaluate

$$(f+g)(4), \quad (f-g)(-2), \quad (fg)(1), \quad \text{and} \quad \left(\frac{f}{g}\right)(0).$$

(a)



(b)

x	$f(x)$	$g(x)$
-2	-3	undefined
0	1	0
1	3	1
4	9	2

(c) $f(x) = 2x + 1, \quad g(x) = \sqrt{x}$

SOLUTION

(a) From the figure, $f(4) = 9$ and $g(4) = 2$.

$$\begin{aligned} &(f+g)(4) \\ &= f(4) + g(4) && (f+g)(x) = f(x) + g(x) \\ &= 9 + 2 && \text{Substitute.} \\ &= 11 && \text{Add.} \end{aligned}$$

For $(f-g)(-2)$, although $f(-2) = -3$, $g(-2)$ is undefined because -2 is not in the domain of g. Thus $(f-g)(-2)$ is undefined.

The domains of f and g both include 1.

$$\begin{aligned} &(fg)(1) \\ &= f(1) \cdot g(1) && (fg)(x) = f(x) \cdot g(x) \\ &= 3 \cdot 1 && \text{Substitute.} \\ &= 3 && \text{Multiply.} \end{aligned}$$

The graph of g includes the origin, so $g(0) = 0$. Thus $\left(\frac{f}{g}\right)(0)$ is undefined.

(b) From the table, $f(4) = 9$ and $g(4) = 2$.

$$\begin{aligned} &(f+g)(4) \\ &= f(4) + g(4) && (f+g)(x) = f(x) + g(x) \\ &= 9 + 2 && \text{Substitute.} \\ &= 11 && \text{Add.} \end{aligned}$$

In the table, $g(-2)$ is undefined, and thus $(f-g)(-2)$ is also undefined.

$$\begin{aligned} &(fg)(1) \\ &= f(1) \cdot g(1) && (fg)(x) = f(x) \cdot g(x) \\ &= 3 \cdot 1 && f(1) = 3 \text{ and } g(1) = 1 \\ &= 3 && \text{Multiply.} \end{aligned}$$

The quotient function value $\left(\frac{f}{g}\right)(0)$ is undefined because the denominator, $g(0)$, equals 0.

(c) Using $f(x) = 2x + 1$ and $g(x) = \sqrt{x}$, we can find $(f + g)(4)$ and $(fg)(1)$. Because -2 is not in the domain of g, $(f - g)(-2)$ is not defined.

$(f + g)(4)$	$(fg)(1)$
$= f(4) + g(4)$	$= f(1) \cdot g(1)$
$= (2 \cdot 4 + 1) + \sqrt{4}$	$= (2 \cdot 1 + 1) \cdot \sqrt{1}$
$= 9 + 2$	$= 3(1)$
$= 11$	$= 3$

$\left(\frac{f}{g}\right)(0)$ is undefined since $g(0) = 0$.

✔ Now Try Exercises 33 and 37.

The Difference Quotient Suppose a point P lies on the graph of $y = f(x)$ as in **Figure 93**, and suppose h is a positive number. If we let $(x, f(x))$ denote the coordinates of P and $(x + h, f(x + h))$ denote the coordinates of Q, then the line joining P and Q has slope as follows.

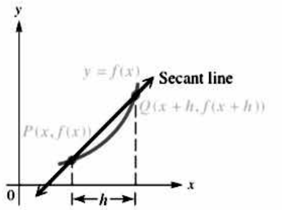


Figure 93

$$m = \frac{f(x + h) - f(x)}{(x + h) - x} \qquad \text{Slope formula}$$

$$m = \frac{\boldsymbol{f(x + h) - f(x)}}{\boldsymbol{h}}, \quad \boldsymbol{h \neq 0} \qquad \text{Difference quotient}$$

This boldface expression is the **difference quotient.**

Figure 93 shows the graph of the line PQ (called a **secant line**). As h approaches 0, the slope of this secant line approaches the slope of the line tangent to the curve at P. Important applications of this idea are developed in calculus.

EXAMPLE 4 Finding the Difference Quotient

Let $f(x) = 2x^2 - 3x$. Find and simplify the expression for the difference quotient,

$$\frac{f(x + h) - f(x)}{h}.$$

SOLUTION We use a three-step process.

Step 1 Find the first term in the numerator, $f(x + h)$. Replace x in $f(x)$ with $x + h$.

$$\begin{aligned} &f(x + h) \\ &= 2(x + h)^2 - 3(x + h) \qquad f(x) = 2x^2 - 3x \end{aligned}$$

Step 2 Find the entire numerator, $f(x + h) - f(x)$.

From Step 1

$$\begin{aligned} &f(x + h) - f(x) \\ &= [2(x + h)^2 - 3(x + h)] - (2x^2 - 3x) && \text{Substitute.} \\ &= 2(x^2 + 2xh + h^2) - 3(x + h) - (2x^2 - 3x) && \text{Square } x + h. \\ &= 2x^2 + 4xh + 2h^2 - 3x - 3h - 2x^2 + 3x && \text{Distributive property} \\ &= 4xh + 2h^2 - 3h && \text{Combine like terms.} \end{aligned}$$

Remember this term when squaring $x + h$

Step 3 Find the difference quotient by dividing by h.

$$\frac{f(x+h)-f(x)}{h}$$

$$= \frac{4xh + 2h^2 - 3h}{h} \quad \text{Substitute } 4xh + 2h^2 - 3h \text{ for } f(x+h) - f(x) \text{, from Step 2.}$$

$$= \frac{h(4x + 2h - 3)}{h} \quad \text{Factor out } h.$$

$$= 4x + 2h - 3 \quad \text{Divide out the common factor.}$$

Now Try Exercises 45 and 55.

CAUTION In **Example 4,** notice that the expression $f(x+h)$ is not equivalent to $f(x) + f(h)$. These expressions differ by $4xh$.

$$f(x+h) = 2(x+h)^2 - 3(x+h) = 2x^2 + 4xh + 2h^2 - 3x - 3h$$

$$f(x) + f(h) = (2x^2 - 3x) + (2h^2 - 3h) = 2x^2 - 3x + 2h^2 - 3h$$

In general, for a function f, $f(x+h)$ is *not* equivalent to $f(x) + f(h)$.

LOOKING AHEAD TO CALCULUS

The difference quotient is essential in the definition of the **derivative of a function** in calculus. The derivative provides a formula, in function form, for finding the slope of the tangent line to the graph of the function at a given point.

To illustrate, it is shown in calculus that the derivative of

$$f(x) = x^2 + 3$$

is given by the function

$$f'(x) = 2x.$$

Now, $f'(0) = 2(0) = 0$, meaning that the slope of the tangent line to $f(x) = x^2 + 3$ at $x = 0$ is 0, which implies that the tangent line is horizontal. If you draw this tangent line, you will see that it is the line $y = 3$, which is indeed a horizontal line.

Composition of Functions and Domain The diagram in **Figure 94** shows a function g that assigns to each x in its domain a value $g(x)$. Then another function f assigns to each $g(x)$ in its domain a value $f(g(x))$. This two-step process takes an element x and produces a corresponding element $f(g(x))$.

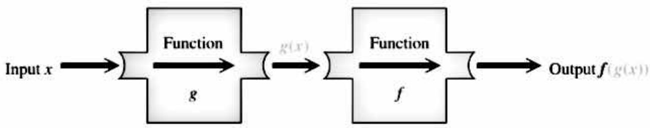


Figure 94

The function with y-values $f(g(x))$ is the *composition* of functions f and g, which is written $\boldsymbol{f \circ g}$ and read **"f of g"** or **"f compose g"**.

Composition of Functions and Domain

If f and g are functions, then the **composite function,** or **composition,** of f and g is defined by

$$(f \circ g)(x) = f(g(x)).$$

The **domain of $f \circ g$** is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f.

As a real-life example of how composite functions occur, consider the following retail situation.

A \$40 pair of blue jeans is on sale for 25% off. If we purchase the jeans before noon, they are an additional 10% off. What is the final sale price of the jeans?

We might be tempted to say that the jeans are 35% off and calculate $\$40(0.35) = \$14$, giving a final sale price of

$$\$40 - \$14 = \$26. \quad \text{Incorrect}$$

\$26 *is not correct.* To find the final sale price, we must first find the price after taking 25% off and then take an additional 10% off *that* price. See **Figure 95.**

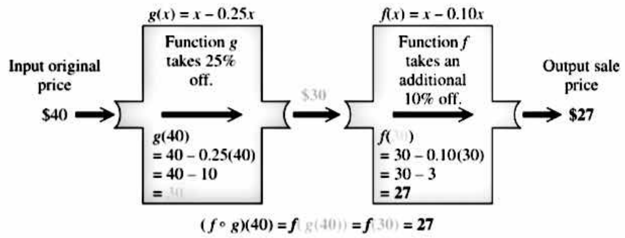


Figure 95

EXAMPLE 5 Evaluating Composite Functions

Let $f(x) = 2x - 1$ and $g(x) = \frac{4}{x-1}$.

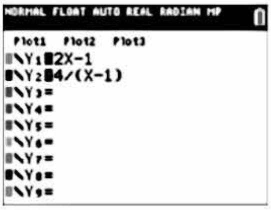
(a) Find $(f \circ g)(2)$. **(b)** Find $(g \circ f)(-3)$.

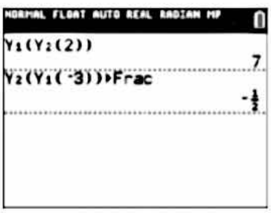
SOLUTION

(a) First find $g(2)$: $g(2) = \frac{4}{2-1} = \frac{4}{1} = 4$.

Now find $(f \circ g)(2)$.

$$\begin{aligned}&(f \circ g)(2)\\ &= f(g(2)) && \text{Definition of composition}\\ &= f(4) && g(2) = 4\\ &= 2(4) - 1 && \text{Definition of } f\\ &= 7 && \text{Simplify.}\end{aligned}$$





The screens show how a graphing calculator evaluates the expressions in **Example 5.**

(b) $(g \circ f)(-3)$

$$\begin{aligned}&= g(f(-3)) && \text{Definition of composition}\\ &= g[2(-3) - 1] && f(-3) = 2(-3) - 1\\ &= g(-7) && \text{Multiply, and then subtract.}\\ &= \frac{4}{-7-1} && g(x) = \frac{4}{x-1}\\ &= -\frac{1}{2} && \text{Subtract in the denominator. Write in lowest terms.}\end{aligned}$$

✔ **Now Try Exercise 57.**

EXAMPLE 6 Determining Composite Functions and Their Domains

Given that $f(x) = \sqrt{x}$ and $g(x) = 4x + 2$, find each of the following.

(a) $(f \circ g)(x)$ and its domain **(b)** $(g \circ f)(x)$ and its domain

SOLUTION

(a) $(f \circ g)(x)$

$$\begin{aligned}&= f(g(x)) && \text{Definition of composition}\\ &= f(4x + 2) && g(x) = 4x + 2\\ &= \sqrt{4x + 2} && f(x) = \sqrt{x}\end{aligned}$$

The domain and range of g are both the set of all real numbers, $(-\infty, \infty)$. The domain of f is the set of all nonnegative real numbers, $[0, \infty)$. Thus, $g(x)$, which is defined as $4x + 2$, must be greater than or equal to zero.

The radicand must be nonnegative.

$$4x + 2 \geq 0 \quad \text{Solve the inequality.}$$

$$x \geq -\frac{1}{2} \quad \text{Subtract 2. Divide by 4.}$$

Therefore, the domain of $f \circ g$ is $\left[-\frac{1}{2}, \infty\right)$.

(b) $(g \circ f)(x)$

$= g(f(x))$ Definition of composition

$= g\left(\sqrt{x}\right)$ $f(x) = \sqrt{x}$

$= 4\sqrt{x} + 2$ $g(x) = 4x + 2$

The domain and range of f are both the set of all nonnegative real numbers, $[0, \infty)$. The domain of g is the set of all real numbers, $(-\infty, \infty)$. Therefore, the domain of $g \circ f$ is $[0, \infty)$.

✔ **Now Try Exercise 75.**

EXAMPLE 7 Determining Composite Functions and Their Domains

Given that $f(x) = \frac{6}{x-3}$ and $g(x) = \frac{1}{x}$, find each of the following.

(a) $(f \circ g)(x)$ and its domain

(b) $(g \circ f)(x)$ and its domain

SOLUTION

(a) $(f \circ g)(x)$

$= f(g(x))$ By definition

$= f\left(\frac{1}{x}\right)$ $g(x) = \frac{1}{x}$

$= \dfrac{6}{\frac{1}{x} - 3}$ $f(x) = \frac{6}{x-3}$

$= \dfrac{6x}{1 - 3x}$ Multiply the numerator and denominator by x.

The domain of g is the set of all real numbers *except* 0, which makes $g(x)$ undefined. The domain of f is the set of all real numbers *except* 3. The expression for $g(x)$, therefore, cannot equal 3. We determine the value that makes $g(x) = 3$ and *exclude* it from the domain of $f \circ g$.

$$\frac{1}{x} = 3 \quad \text{The solution must be excluded.}$$

$$1 = 3x \quad \text{Multiply by } x.$$

$$x = \frac{1}{3} \quad \text{Divide by 3.}$$

Therefore, the domain of $f \circ g$ is the set of all real numbers *except* 0 and $\frac{1}{3}$, written in interval notation as

$$(-\infty, 0) \cup \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right).$$

LOOKING AHEAD TO CALCULUS
Finding the derivative of a function in calculus is called **differentiation.** To differentiate a composite function such as

$$h(x) = (3x + 2)^4,$$

we interpret $h(x)$ as $(f \circ g)(x)$, where

$$g(x) = 3x + 2 \quad \text{and} \quad f(x) = x^4.$$

The **chain rule** allows us to differentiate composite functions. Notice the use of the composition symbol and function notation in the following, which comes from the chain rule.

If $h(x) = (f \circ g)(x)$, then
$h'(x) = f'(g(x)) \cdot g'(x)$.

(b) $(g \circ f)(x)$

$$= g(f(x)) \quad \text{By definition}$$

$$= g\left(\frac{6}{x-3}\right) \quad f(x) = \frac{6}{x-3}$$

$$= \frac{1}{\frac{6}{x-3}} \quad \text{Note that this is meaningless if } x = 3;\ g(x) = \tfrac{1}{x}.$$

$$= \frac{x-3}{6} \quad \frac{1}{\frac{a}{b}} = 1 \div \frac{a}{b} = 1 \cdot \frac{b}{a} = \frac{b}{a}$$

The domain of f is the set of all real numbers *except* 3. The domain of g is the set of all real numbers *except* 0. The expression for $f(x)$, which is $\frac{6}{x-3}$, is never zero because the numerator is the nonzero number 6. The domain of $g \circ f$ is the set of all real numbers *except* 3, written $(-\infty, 3) \cup (3, \infty)$.

✔ **Now Try Exercise 87.**

NOTE It often helps to consider the *unsimplified* form of a composition expression when determining the domain in a situation like **Example 7(b).**

EXAMPLE 8 Showing That $(g \circ f)(x)$ Is Not Equivalent to $(f \circ g)(x)$

Let $f(x) = 4x + 1$ and $g(x) = 2x^2 + 5x$. Show that $(g \circ f)(x) \neq (f \circ g)(x)$. (This is sufficient to prove that this inequality is true in general.)

SOLUTION First, find $(g \circ f)(x)$. Then find $(f \circ g)(x)$.

$(g \circ f)(x)$

$$= g(f(x)) \quad \text{By definition}$$
$$= g(4x + 1) \quad f(x) = 4x + 1$$
$$= 2(4x + 1)^2 + 5(4x + 1) \quad g(x) = 2x^2 + 5x$$
$$= 2(16x^2 + 8x + 1) + 20x + 5 \quad \text{Square } 4x + 1 \text{ and apply the distributive property.}$$
$$= 32x^2 + 16x + 2 + 20x + 5 \quad \text{Distributive property}$$
$$= 32x^2 + 36x + 7 \quad \text{Combine like terms.}$$

$(f \circ g)(x)$

$$= f(g(x)) \quad \text{By definition}$$
$$= f(2x^2 + 5x) \quad g(x) = 2x^2 + 5x$$
$$= 4(2x^2 + 5x) + 1 \quad f(x) = 4x + 1$$
$$= 8x^2 + 20x + 1 \quad \text{Distributive property}$$

Thus, $(g \circ f)(x) \neq (f \circ g)(x)$.

✔ **Now Try Exercise 91.**

As **Example 8** shows, ***it is not always true that*** $f \circ g = g \circ f$. One important circumstance in which equality holds occurs when f and g are *inverses* of each other, a concept discussed later in the text.