ALGEBRAIC SOLUTION

(d) To make a profit, $P(x)$ must be positive.

$P(x) = 25x - 1500$ Profit function from part (c)

Set $P(x) > 0$ and solve.

$$P(x) > 0$$
$$25x - 1500 > 0 \quad P(x) = 25x - 1500$$
$$25x > 1500 \quad \text{Add 1500 to each side.}$$
$$x > 60 \quad \text{Divide by 25.}$$

The number of items must be a whole number, so at least 61 items must be sold for the company to make a profit.

GRAPHING CALCULATOR SOLUTION

(d) Define y_1 as $25x - 1500$ and graph the line. Use the capability of a calculator to locate the x-intercept. See **Figure 42.** As the graph shows, y-values for x less than 60 are negative, and y-values for x greater than 60 are positive, so at least 61 items must be sold for the company to make a profit.

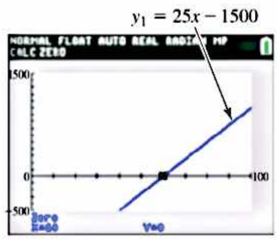


Figure 42

Now Try Exercise 77.

CAUTION In problems involving $R(x) - C(x)$ like **Example 9(c),** pay attention to the use of parentheses around the expression for $C(x)$.

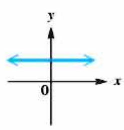
2.4 Exercises

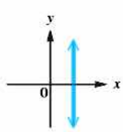
CONCEPT PREVIEW *Match the description in Column I with the correct response in Column II. Some choices may not be used.*

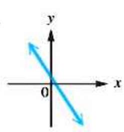
I	II
1. a linear function whose graph has y-intercept $(0, 6)$	**A.** $f(x) = 5x$
2. a vertical line	**B.** $f(x) = 3x + 6$
3. a constant function	**C.** $f(x) = -8$
4. a linear function whose graph has x-intercept $(-2, 0)$ and y-intercept $(0, 4)$	**D.** $f(x) = x^2$
	E. $x + y = -6$
5. a linear function whose graph passes through the origin	**F.** $f(x) = 3x + 4$
	G. $2x - y = -4$
6. a function that is not linear	**H.** $x = 9$

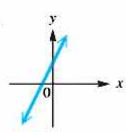
CONCEPT PREVIEW *For each given slope, identify the line in A–D that could have this slope.*

7. -3 **8.** 0 **9.** 3 **10.** undefined

A.


B.


C.


D.


Graph each linear function. Give the domain and range. Identify any constant functions. ***See Examples 1 and 2.***

11. $f(x) = x - 4$ **12.** $f(x) = -x + 4$ **13.** $f(x) = \frac{1}{2}x - 6$

14. $f(x) = \frac{2}{3}x + 2$ **15.** $f(x) = 3x$ **16.** $f(x) = -2x$

17. $f(x) = -4$ **18.** $f(x) = 3$ **19.** $f(x) = 0$

20. *Concept Check* Write the equation of the linear function f with graph having slope 9 and passing through the origin. Give the domain and range.

Graph each line. Give the domain and range. ***See Examples 3 and 4.***

21. $-4x + 3y = 12$ **22.** $2x + 5y = 10$ **23.** $3y - 4x = 0$

24. $3x + 2y = 0$ **25.** $x = 3$ **26.** $x = -4$

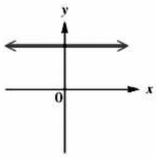
27. $2x + 4 = 0$ **28.** $-3x + 6 = 0$

29. $-x + 5 = 0$ **30.** $3 + x = 0$

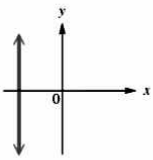
Match each equation with the sketch that most closely resembles its graph. ***See Examples 2 and 3.***

31. $y = 5$ **32.** $y = -5$ **33.** $x = 5$ **34.** $x = -5$

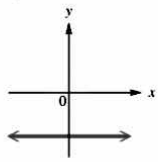
A.



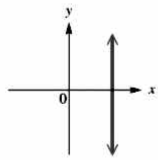
B.



C.



D.



Use a graphing calculator to graph each equation in the standard viewing window. ***See Examples 1, 2, and 4.***

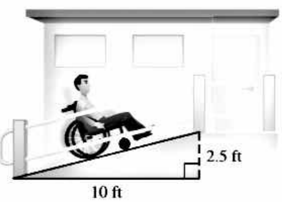
35. $y = 3x + 4$ **36.** $y = -2x + 3$

37. $3x + 4y = 6$ **38.** $-2x + 5y = 10$

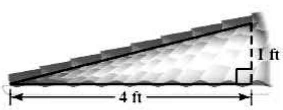
39. *Concept Check* If a walkway rises 2.5 ft for every 10 ft on the horizontal, which of the following express its slope (or grade)? (There are several correct choices.)

A. 0.25 **B.** 4 **C.** $\frac{2.5}{10}$ **D.** 25%

E. $\frac{1}{4}$ **F.** $\frac{10}{2.5}$ **G.** 400% **H.** 2.5%



40. *Concept Check* If the pitch of a roof is $\frac{1}{4}$, how many feet in the horizontal direction correspond to a rise of 4 ft?



Find the slope of the line satisfying the given conditions. ***See Example 5.***

41. through $(2, -1)$ and $(-3, -3)$

42. through $(-3, 4)$ and $(2, -8)$

43. through $(5, 8)$ and $(3, 12)$

44. through $(5, -3)$ and $(1, -7)$

45. through $(5, 9)$ and $(-2, 9)$

46. through $(-2, 4)$ and $(6, 4)$

47. horizontal, through $(5, 1)$

48. horizontal, through $(3, 5)$

49. vertical, through $(4, -7)$

50. vertical, through $(-8, 5)$

For each line, ***(a)*** *find the slope and* ***(b)*** *sketch the graph.* ***See Examples 6 and 7.***

51. $y = 3x + 5$

52. $y = 2x - 4$

53. $2y = -3x$

54. $-4y = 5x$

55. $5x - 2y = 10$

56. $4x + 3y = 12$

Graph the line passing through the given point and having the indicated slope. Plot two points on the line. ***See Example 7.***

57. through $(-1, 3)$, $m = \frac{3}{2}$

58. through $(-2, 8)$, $m = \frac{2}{5}$

59. through $(3, -4)$, $m = -\frac{1}{3}$

60. through $(-2, -3)$, $m = -\frac{3}{4}$

61. through $\left(-\frac{1}{2}, 4\right)$, $m = 0$

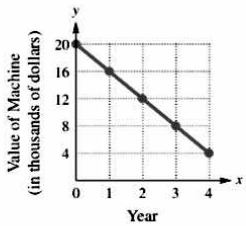
62. through $\left(\frac{3}{2}, 2\right)$, $m = 0$

63. through $\left(-\frac{5}{2}, 3\right)$, undefined slope

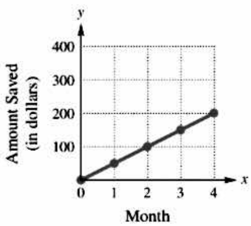
64. through $\left(\frac{9}{4}, 2\right)$, undefined slope

Concept Check *Find and interpret the average rate of change illustrated in each graph.*

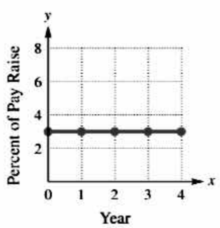
65.



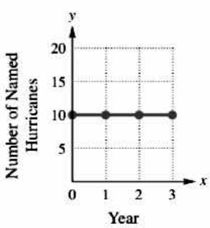
66.



67.



68.



Solve each problem. ***See Example 8.***

69. ***Dropouts*** In 1980, the number of high school dropouts in the United States was 5085 thousand. By 2012, this number had decreased to 2562 thousand. Find and interpret the average rate of change per year in the number of high school dropouts. Round the answer to the nearest tenth. (*Source: 2013 Digest of Education Statistics.*)

70. ***Plasma Flat-Panel TV Sales*** The total amount spent on plasma flat-panel TVs in the United States changed from \$5302 million in 2006 to \$1709 million in 2013. Find and interpret the average rate of change in sales, in millions of dollars per year. Round the answer to the nearest hundredth. (*Source:* Consumer Electronics Association.)

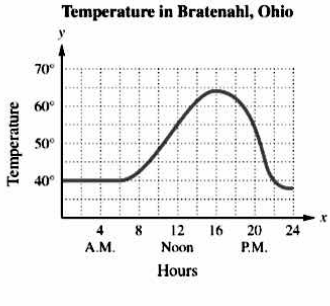
95. ***Temperature*** The graph shows temperatures on a given day in Bratenahl, Ohio.

(a) At what times during the day was the temperature over 55°?

(b) When was the temperature at or below 40°?

(c) Greenville, South Carolina, is 500 mi south of Bratenahl, Ohio, and its temperature is 7° higher all day long. At what time was the temperature in Greenville the same as the temperature at noon in Bratenahl?

(d) Use the graph to give a word description of the 24-hr period in Bratenahl.

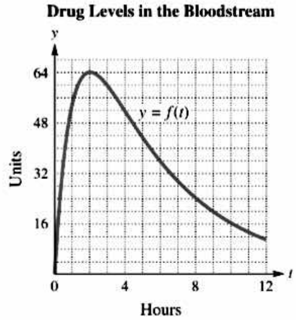


96. ***Drug Levels in the Bloodstream*** When a drug is taken orally, the amount of the drug in the bloodstream after t hours is given by the function $y = f(t)$, as shown in the graph.

(a) How many units of the drug are in the bloodstream at 8 hr?

(b) During what time interval is the level of the drug in the bloodstream increasing? decreasing?

(c) When does the level of the drug in the bloodstream reach its maximum value, and how many units are in the bloodstream at that time?



(d) When the drug reaches its maximum level in the bloodstream, how many additional hours are required for the level to drop to 16 units?

(e) Use the graph to give a word description of the 12-hr period.

2.4 Linear Functions

- **Basic Concepts of Linear Functions**
- **Standard Form $Ax + By = C$**
- **Slope**
- **Average Rate of Change**
- **Linear Models**

Basic Concepts of Linear Functions

We begin our study of specific functions by looking at *linear* functions.

Linear Function

A function f is a **linear function** if, for real numbers a and b,

$$f(x) = ax + b.$$

If $a \neq 0$, then the domain and the range of f are both $(-\infty, \infty)$.

Lines can be graphed by finding ordered pairs and plotting them. Although only two points are necessary to graph a linear function, we usually plot a third point as a check. The intercepts are often good points to choose for graphing lines.

EXAMPLE 1 Graphing a Linear Function Using Intercepts

Graph $f(x) = -2x + 6$. Give the domain and range.

SOLUTION The x-intercept is found by letting $f(x) = 0$ and solving for x.

$$f(x) = -2x + 6$$
$$0 = -2x + 6 \quad \text{Let } f(x) = 0.$$
$$x = 3 \quad \text{Add } 2x \text{ and divide by 2.}$$

We plot the x-intercept $(3, 0)$. The y-intercept is found by evaluating $f(0)$.

$$f(0) = -2(0) + 6 \quad \text{Let } x = 0.$$
$$f(0) = 6 \quad \text{Simplify.}$$

Therefore, another point on the graph is the y-intercept, $(0, 6)$. We plot this point and join the two points with a straight-line graph. We use the point $(2, 2)$ as a check. See **Figure 31.** The domain and the range are both $(-\infty, \infty)$.

The corresponding calculator graph with $f(x) = y_1$ is shown in **Figure 32.**

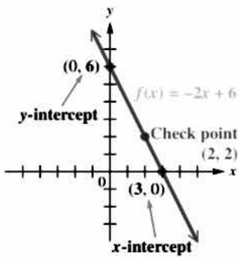


Figure 31

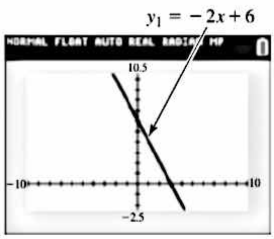


Figure 32

✔ **Now Try Exercise 13.**

If $a = 0$ in the definition of linear function, then the equation becomes $f(x) = b$. In this case, the domain is $(-\infty, \infty)$ and the range is $\{b\}$. A function of the form $f(x) = b$ is a **constant function,** and its graph is a horizontal line.

EXAMPLE 2 Graphing a Horizontal Line

Graph $f(x) = -3$. Give the domain and range.

SOLUTION Because $f(x)$, or y, always equals -3, the value of y can never be 0 and the graph has no x-intercept. If a straight line has no x-intercept then it must be parallel to the x-axis, as shown in **Figure 33.** The domain of this linear function is $(-\infty, \infty)$ and the range is $\{-3\}$. **Figure 34** shows the calculator graph.

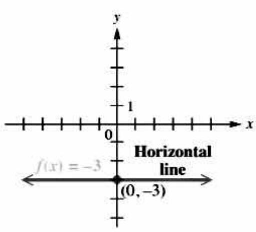


Figure 33

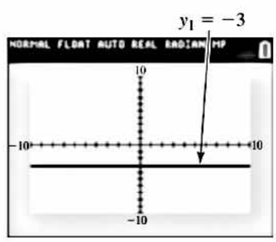


Figure 34

✔ **Now Try Exercise 17.**

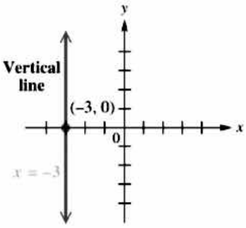


Figure 35

EXAMPLE 3 **Graphing a Vertical Line**

Graph $x = -3$. Give the domain and range of this relation.

SOLUTION Because x always equals -3, the value of x can never be 0, and the graph has no y-intercept. Using reasoning similar to that of **Example 2,** we find that this graph is parallel to the y-axis, as shown in **Figure 35.** The domain of this relation, which is *not* a function, is $\{-3\}$, while the range is $(-\infty, \infty)$.

✔ Now Try Exercise 25.

Standard Form $Ax + By = C$ Equations of lines are often written in the form $Ax + By = C$, known as **standard form.**

NOTE The definition of "standard form" is, ironically, not standard from one text to another. Any linear equation can be written in infinitely many different, but equivalent, forms. For example, the equation $2x + 3y = 8$ can be written equivalently as

$$2x + 3y - 8 = 0, \quad 3y = 8 - 2x, \quad x + \frac{3}{2}y = 4, \quad 4x + 6y = 16,$$

and so on. In this text we will agree that if the coefficients and constant in a linear equation are rational numbers, then we will consider the standard form to be $Ax + By = C$, where $A \geq 0$, A, B, and C are integers, and the greatest common factor of A, B, and C is 1. If $A = 0$, then we choose $B > 0$. (If two or more integers have a greatest common factor of 1, they are said to be **relatively prime.**)

EXAMPLE 4 **Graphing $Ax + By = C$ ($C = 0$)**

Graph $4x - 5y = 0$. Give the domain and range.

SOLUTION Find the intercepts.

$4x - 5y = 0$

$4(0) - 5y = 0$ Let $x = 0$.

$y = 0$ The y-intercept is $(0, 0)$.

$4x - 5y = 0$

$4x - 5(0) = 0$ Let $y = 0$.

$x = 0$ The x-intercept is $(0, 0)$.

The graph of this function has just one intercept—the origin $(0, 0)$. We need to find an additional point to graph the function by choosing a different value for x (or y).

$4(5) - 5y = 0$ We choose $x = 5$.

$20 - 5y = 0$ Multiply.

$4 = y$ Add $5y$. Divide by 5.

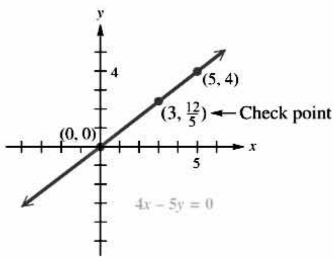


Figure 36

This leads to the ordered pair $(5, 4)$. Complete the graph using the two points $(0, 0)$ and $(5, 4)$, with a third point as a check. The domain and range are both $(-\infty, \infty)$. See **Figure 36.**

✔ Now Try Exercise 23.

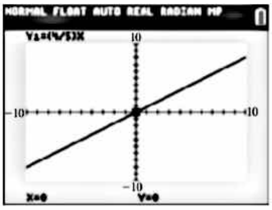


Figure 37

To use a graphing calculator to graph a linear function, as in **Figure 37,** we must first solve the defining equation for y.

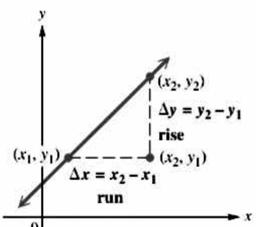
$$4x - 5y = 0 \quad \text{Equation from Example 4}$$

$$y = \frac{4}{5}x \quad \text{Subtract } 4x\text{. Divide by } -5.$$

■

Slope Slope is a numerical measure of the steepness and orientation of a straight line. (Geometrically, this may be interpreted as the ratio of **rise** to **run.**) The slope of a highway (sometimes called the *grade*) is often given as a percent. For example, a 10% $\left(\text{or } \frac{10}{100} = \frac{1}{10}\right)$ slope means the highway rises 1 unit for every 10 horizontal units.

To find the slope of a line, start with two distinct points (x_1, y_1) and (x_2, y_2) on the line, as shown in **Figure 38,** where $x_1 \neq x_2$. As we move along the line from (x_1, y_1) to (x_2, y_2), the horizontal difference



$\text{Slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \quad (\Delta x \neq 0)$

Figure 38

$$\Delta x = x_2 - x_1$$

is the **change in x,** denoted by Δx (read **"delta x"**), where Δ is the Greek capital letter **delta.** The vertical difference, the **change in y,** can be written

$$\Delta y = y_2 - y_1.$$

The *slope* of a nonvertical line is defined as the quotient (ratio) of the change in y and the change in x, as follows.

Slope

The **slope** m of the line through the points (x_1, y_1) and (x_2, y_2) is given by the following.

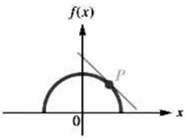
$$m = \frac{\textbf{rise}}{\textbf{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}, \quad \textbf{where } \Delta x \neq 0$$

***That is, the slope of a line is the change in y divided by the corresponding change in x, where the change in x is not* 0.**

LOOKING AHEAD TO CALCULUS

The concept of slope of a line is extended in calculus to general curves. The **slope of a curve at a point** is understood to mean the slope of the line tangent to the curve at that point.

The line in the figure is tangent to the curve at point P.



CAUTION ***When using the slope formula, it makes no difference which point is (x_1, y_1) or (x_2, y_2). However, be consistent.*** Start with the x- and y-values of *one* point (either one), and subtract the corresponding values of the *other* point.

$$\textit{Use} \quad \frac{y_2 - y_1}{x_2 - x_1} \quad \text{or} \quad \frac{y_1 - y_2}{x_1 - x_2}, \quad \textit{not} \quad \frac{y_2 - y_1}{x_1 - x_2} \quad \text{or} \quad \frac{y_1 - y_2}{x_2 - x_1}.$$

Be sure to write the difference of the y-values in the numerator and the difference of the x-values in the denominator.

The slope of a line can be found only if the line is nonvertical. This guarantees that $x_2 \neq x_1$ so that the denominator $x_2 - x_1 \neq 0$.

Undefined Slope

The slope of a vertical line is undefined.

EXAMPLE 5 Finding Slopes with the Slope Formula

Find the slope of the line through the given points.

(a) $(-4, 8), (2, -3)$ **(b)** $(2, 7), (2, -4)$ **(c)** $(5, -3), (-2, -3)$

SOLUTION

(a) Let $x_1 = -4$, $y_1 = 8$, and $x_2 = 2$, $y_2 = -3$.

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} \qquad \text{Definition of slope}$$

$$= \frac{-3 - 8}{2 - (-4)} \qquad \text{Substitute carefully.}$$

$$= \frac{-11}{6}, \text{ or } -\frac{11}{6} \qquad \text{Subtract; } \frac{-a}{b} = -\frac{a}{b}.$$

We can also subtract in the opposite order, letting $x_1 = 2$, $y_1 = -3$ and $x_2 = -4$, $y_2 = 8$. The same slope results.

$$m = \frac{8 - (-3)}{-4 - 2} = \frac{11}{-6}, \text{ or } -\frac{11}{6}$$

(b) If we attempt to use the slope formula with the points $(2, 7)$ and $(2, -4)$, we obtain a zero denominator.

$$m = \frac{-4 - 7}{2 - 2} = \frac{-11}{0} \qquad \textbf{Undefined}$$

The formula is not valid here because $\Delta x = x_2 - x_1 = 2 - 2 = 0$. A sketch would show that the line through $(2, 7)$ and $(2, -4)$ is vertical. As mentioned above, the slope of a vertical line is undefined.

(c) For $(5, -3)$ and $(-2, -3)$, the slope equals 0.

$$m = \frac{-3 - (-3)}{-2 - 5} = \frac{0}{-7} = 0$$

A sketch would show that the line through $(5, -3)$ and $(-2, -3)$ is horizontal.

✔ **Now Try Exercises 41, 47, and 49.**

LOOKING AHEAD TO CALCULUS

The **derivative** of a function provides a formula for determining the slope of a line tangent to a curve. If the slope is positive on a given interval, then the function is increasing there. If it is negative, then the function is decreasing. If it is 0, then the function is constant.

The results in **Example 5(c)** suggest the following generalization.

Slope Equal to Zero

The slope of a horizontal line is 0.

Theorems for similar triangles can be used to show that the slope of a line is independent of the choice of points on the line. ***That is, slope is the same no matter which pair of distinct points on the line are used to find it.***

If the equation of a line is in the form

$$y = ax + b,$$

we can show that the slope of the line is a. To do this, we use function notation and the definition of slope.

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \qquad \text{Slope formula}$$

$$m = \frac{[a(x+1) + b] - (ax + b)}{(x+1) - x} \qquad \text{Let } f(x) = ax + b,\ x_1 = x, \text{ and } x_2 = x + 1.$$

$$m = \frac{ax + a + b - ax - b}{x + 1 - x} \qquad \text{Distributive property}$$

$$m = \frac{a}{1} \qquad \text{Combine like terms.}$$

$$m = a \qquad \text{The slope is } a.$$

This discussion enables us to find the slope of the graph of any linear equation by solving for y and identifying the coefficient of x, which is the slope.

EXAMPLE 6 Finding Slope from an Equation

Find the slope of the line $4x + 3y = 12$.

SOLUTION Solve the equation for y.

$$4x + 3y = 12$$

$$3y = -4x + 12 \qquad \text{Subtract } 4x.$$

Be careful to divide *each* term by 3.

$$y = -\frac{4}{3}x + 4 \qquad \text{Divide by 3.}$$

The slope is $-\frac{4}{3}$, which is the coefficient of x when the equation is solved for y.

Now Try Exercise 55(a).

Because the slope of a line is the ratio of vertical change (rise) to horizontal change (run), if we know the slope of a line and the coordinates of a point on the line, we can draw the graph of the line.

EXAMPLE 7 Graphing a Line Using a Point and the Slope

Graph the line passing through the point $(-1, 5)$ and having slope $-\frac{5}{3}$.

SOLUTION First locate the point $(-1, 5)$ as shown in **Figure 39.** The slope of this line is $\frac{-5}{3}$, so a change of -5 units vertically (that is, 5 units *down*) corresponds to a change of 3 units horizontally (that is, 3 units to the *right*). This gives a second point, $(2, 0)$, which can then be used to complete the graph.

Because $\frac{-5}{3} = \frac{5}{-3}$, another point could be obtained by starting at $(-1, 5)$ and moving 5 units *up* and 3 units to the *left*. We would reach a different second point, $(-4, 10)$, but the graph would be the same line. Confirm this in **Figure 39.**

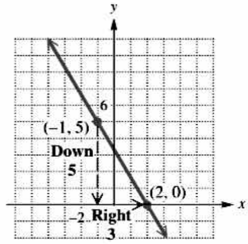


Figure 39

Now Try Exercise 59.

Figure 40 shows lines with various slopes.

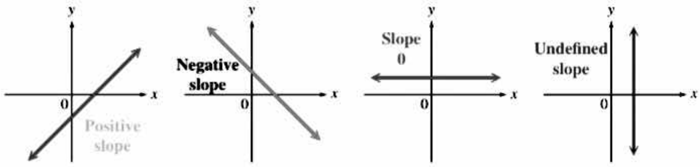


Figure 40

Notice the following important concepts.

- A line with a positive slope rises from left to right. The corresponding linear function is increasing on its entire domain.
- A line with a negative slope falls from left to right. The corresponding linear function is decreasing on its entire domain.
- A line with slope 0 neither rises nor falls. The corresponding linear function is constant on its entire domain.
- The slope of a vertical line is undefined.

Average Rate of Change We know that the slope of a line is the ratio of the vertical change in y to the horizontal change in x. ***Thus, slope gives the average rate of change in y per unit of change in x, where the value of y depends on the value of x.*** If f is a linear function defined on the interval $[a, b]$, then we have the following.

$$\textbf{Average rate of change on } [a, b] = \frac{f(b) - f(a)}{b - a}$$

This is simply another way to write the slope formula, using function notation.

EXAMPLE 8 Interpreting Slope as Average Rate of Change

In 2009, Google spent $2800 million on research and development. In 2013, Google spent $8000 million on research and development. Assume a linear relationship, and find the average rate of change in the amount of money spent on R&D per year. Graph as a line segment, and interpret the result. (*Source:* MIT Technology Review.)

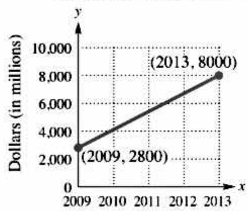


Figure 41

SOLUTION To use the slope formula, we need two ordered pairs. Here, If $x = 2009$, then $y = 2800$, and if $x = 2013$, then $y = 8000$. This gives the two ordered pairs $(2009, 2800)$ and $(2013, 8000)$. (Here y is in millions of dollars.)

$$\text{Average rate of change} = \frac{8000 - 2800}{2013 - 2009} = \frac{5200}{4} = 1300$$

The graph in **Figure 41** confirms that the line through the ordered pairs rises from left to right and therefore has positive slope. Thus, the annual amount of money spent by Google on R&D *increased* by an average of about of $1300 million each year from 2009 to 2013.

Now Try Exercise 75.