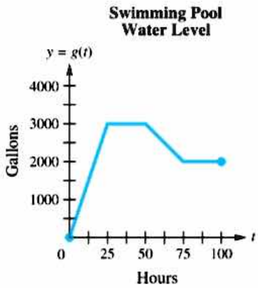


Figure 30 (repeated)

SOLUTION

(a) The maximum range value is 3000, as indicated by the horizontal line segment for the hours 25 to 50. This maximum number of gallons, 3000, is first reached at $t = 25$ hr.

(b) The water level is increasing for $25 - 0 = 25$ hr. The water level is decreasing for $75 - 50 = 25$ hr. It is constant for

$$\begin{aligned}&(50 - 25) + (100 - 75)\\&= 25 + 25\\&= 50 \text{ hr.}\end{aligned}$$

(c) When $t = 90$, $y = g(90) = 2000$. There are 2000 gal after 90 hr.

(d) Looking at the graph in **Figure 30,** we might write the following description.

The pool is empty at the beginning and then is filled to a level of 3000 gal during the first 25 hr. For the next 25 hr, the water level remains the same. At 50 hr, the pool starts to be drained, and this draining lasts for 25 hr, until only 2000 gal remain. For the next 25 hr, the water level is unchanged.

Now Try Exercise 93.

2.3 Exercises

CONCEPT PREVIEW *Fill in the blank(s) to correctly complete each sentence.*

1. The domain of the relation $\{(3, 5), (4, 9), (10, 13)\}$ is ________.

2. The range of the relation in **Exercise 1** is ________.

3. The equation $y = 4x - 6$ defines a function with independent variable ________ and dependent variable ________.

4. The function in **Exercise 3** includes the ordered pair (6, ___).

5. For the function $f(x) = -4x + 2$, $f(-2) =$ ________.

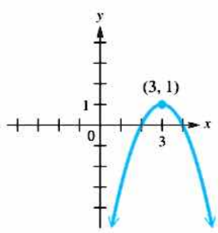
6. For the function $g(x) = \sqrt{x}$, $g(9) =$ ________.

7. The function in **Exercise 6** has domain ________.

8. The function in **Exercise 6** has range ________.

9. The largest open interval over which the function graphed here increases is ________.

10. The largest open interval over which the function graphed here decreases is ________.



Decide whether each relation defines a function. See ***Example 1.***

11. $\{(5, 1), (3, 2), (4, 9), (7, 8)\}$

12. $\{(8, 0), (5, 7), (9, 3), (3, 8)\}$

13. $\{(2, 4), (0, 2), (2, 6)\}$

14. $\{(9, -2), (-3, 5), (9, 1)\}$

15. $\{(-3, 1), (4, 1), (-2, 7)\}$

16. $\{(-12, 5), (-10, 3), (8, 3)\}$

17.

x	y
3	-4
7	-4
10	-4

18.

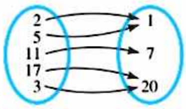
x	y
-4	$\sqrt{2}$
0	$\sqrt{2}$
4	$\sqrt{2}$

Decide whether each relation defines a function, and give the domain and range. ***See Examples 1–4.***

19. $\{(1, 1), (1, -1), (0, 0), (2, 4), (2, -4)\}$

20. $\{(2, 5), (3, 7), (3, 9), (5, 11)\}$

21.



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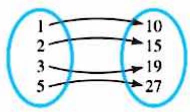


23.

x	y
0	0
-1	1
-2	2

24.

x	y
0	0
1	-1
2	-2

25. ***Number of Visits to U.S. National Parks***

Year (x)	Number of Visits (y) (millions)
2010	64.9
2011	63.0
2012	65.1
2013	63.5

Source: National Park Service.

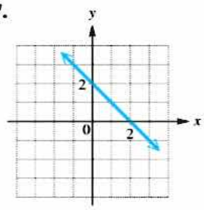
26. ***Attendance at NCAA Women's College Basketball Games***

Season* (x)	Attendance (y)
2011	11,159,999
2012	11,210,832
2013	11,339,285
2014	11,181,735

Source: NCAA.

*Each season overlaps the given year with the previous year.

27.



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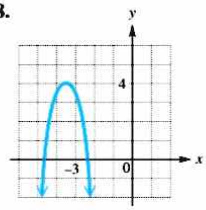

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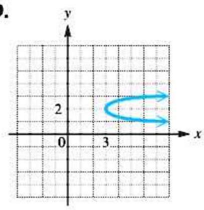

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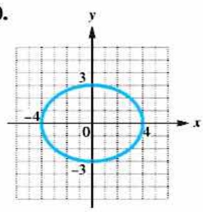

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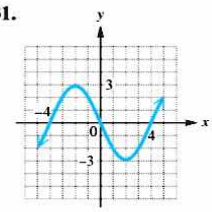

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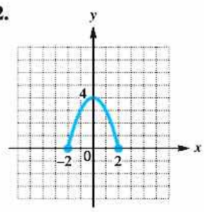


Decide whether each relation defines y as a function of x. Give the domain and range. ***See Example 5.***

33. $y = x^2$ **34.** $y = x^3$ **35.** $x = y^6$

36. $x = y^4$ **37.** $y = 2x - 5$ **38.** $y = -6x + 4$

39. $x + y < 3$ **40.** $x - y < 4$ **41.** $y = \sqrt{x}$

42. $y = -\sqrt{x}$ **43.** $xy = 2$ **44.** $xy = -6$

45. $y = \sqrt{4x + 1}$ **46.** $y = \sqrt{7 - 2x}$

47. $y = \dfrac{2}{x - 3}$ **48.** $y = \dfrac{-7}{x - 5}$

49. *Concept Check* Choose the correct answer: For function f, the notation $f(3)$ means

A. the variable f times 3, or $3f$.

B. the value of the dependent variable when the independent variable is 3.

C. the value of the independent variable when the dependent variable is 3.

D. f equals 3.

50. *Concept Check* Give an example of a function from everyday life. (*Hint:* Fill in the blanks: ______ depends on ______, so ______ is a function of ______.)

Let $f(x) = -3x + 4$ *and* $g(x) = -x^2 + 4x + 1$*. Find each of the following. Simplify if necessary.* ***See Example 6.***

51. $f(0)$ **52.** $f(-3)$ **53.** $g(-2)$

54. $g(10)$ **55.** $f\left(\frac{1}{3}\right)$ **56.** $f\left(-\frac{7}{3}\right)$

57. $g\left(\frac{1}{2}\right)$ **58.** $g\left(-\frac{1}{4}\right)$ **59.** $f(p)$

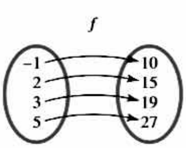
60. $g(k)$ **61.** $f(-x)$ **62.** $g(-x)$

63. $f(x + 2)$ **64.** $f(a + 4)$ **65.** $f(2m - 3)$ **66.** $f(3t - 2)$

For each function, find ***(a)*** $f(2)$ *and* ***(b)*** $f(-1)$*.* ***See Example 7.***

67. $f = \{(-1, 3), (4, 7), (0, 6), (2, 2)\}$ **68.** $f = \{(2, 5), (3, 9), (-1, 11), (5, 3)\}$

69.



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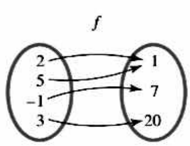

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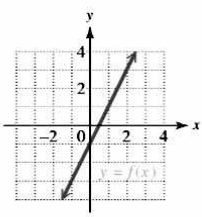

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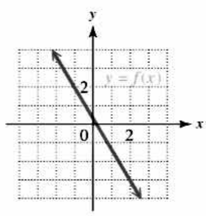
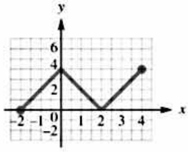


*Use the graph of $y = f(x)$ to find each function value: **(a)** $f(-2)$, **(b)** $f(0)$, **(c)** $f(1)$, and **(d)** $f(4)$. **See Example 7(d).***

73.



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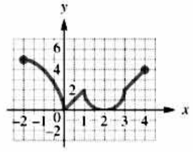

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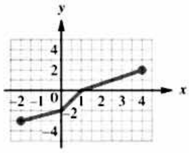

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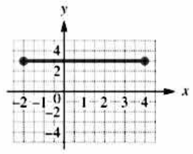


*An equation that defines y as a function of x is given. **(a)** Rewrite each equation using function notation $f(x)$. **(b)** Find $f(3)$. **See Example 8.***

77. $x + 3y = 12$

78. $x - 4y = 8$

79. $y + 2x^2 = 3 - x$

80. $y - 3x^2 = 2 + x$

81. $4x - 3y = 8$

82. $-2x + 5y = 9$

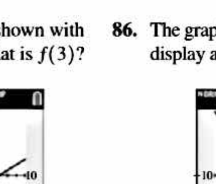
Concept Check Answer each question.

83. If $(3, 4)$ is on the graph of $y = f(x)$, which one of the following must be true: $f(3) = 4$ or $f(4) = 3$?

84. The figure shows a portion of the graph of

$$f(x) = x^2 + 3x + 1$$

and a rectangle with its base on the x-axis and a vertex on the graph. What is the area of the rectangle? (*Hint:* $f(0.2)$ is the height.)

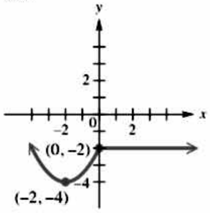


85. The graph of $y_1 = f(x)$ is shown with a display at the bottom. What is $f(3)$?

86. The graph of $y_1 = f(x)$ is shown with a display at the bottom. What is $f(-2)$?

Determine the largest open intervals of the domain over which each function is (a) increasing, (b) decreasing, and (c) constant. See Example 9.

87.



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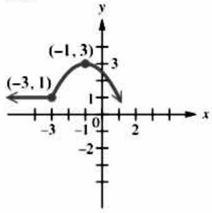

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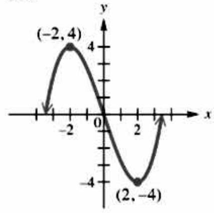

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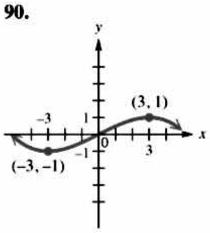

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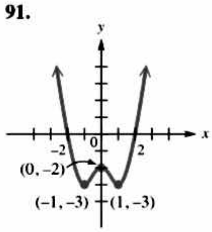

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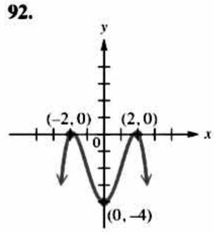


Solve each problem. ***See Example 10.***

93. *Electricity Usage* The graph shows the daily megawatts of electricity used on a record-breaking summer day in Sacramento, California.

(a) Is this the graph of a function?

(b) What is the domain?

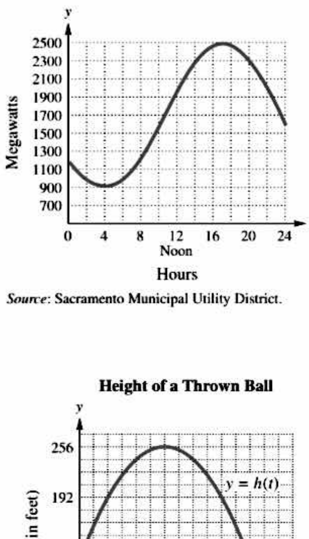
(c) Estimate the number of megawatts used at 8 A.M.

(d) At what time was the most electricity used? the least electricity?

(e) Call this function f. What is $f(12)$? Interpret your answer.

(f) During what time intervals is usage increasing? decreasing?

Electricity Use



Source: Sacramento Municipal Utility District.

94. *Height of a Ball* A ball is thrown straight up into the air. The function $y = h(t)$ in the graph gives the height of the ball (in feet) at t seconds. (*Note:* The graph does *not* show the path of the ball. The ball is rising straight up and then falling straight down.)

(a) What is the height of the ball at 2 sec?

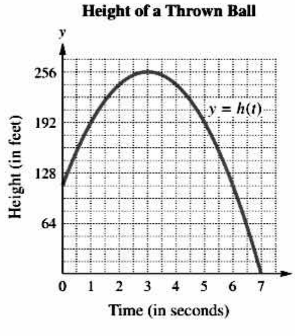
(b) When will the height be 192 ft?

(c) During what time intervals is the ball going up? down?

(d) How high does the ball go? When does the ball reach its maximum height?

(e) After how many seconds does the ball hit the ground?

Height of a Thrown Ball



EXAMPLE 1 Deciding Whether Relations Define Functions

Decide whether each relation defines a function.

$$F = \{(1, 2), (-2, 4), (3, 4)\}$$
$$G = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$$
$$H = \{(-4, 1), (-2, 1), (-2, 0)\}$$

SOLUTION Relation F is a function because for each different x-value there is exactly one y-value. We can show this correspondence as follows.

$$\begin{array}{ll} \{1, -2, 3\} & x\text{-values of } F \\ \downarrow \;\; \downarrow \;\; \downarrow & \\ \{2, \;4, \;4\} & y\text{-values of } F \end{array}$$

As the correspondence below shows, relation G is not a function because one first component corresponds to *more than one* second component.

$$\begin{array}{ll} \{1, 2\} & x\text{-values of } G \\ \{1, 2, 3\} & y\text{-values of } G \end{array}$$

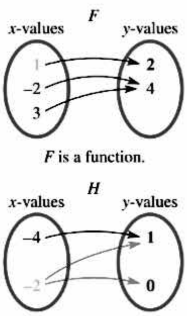
In relation H the last two ordered pairs have the same x-value paired with two different y-values (-2 is paired with both 1 and 0), so H is a relation but not a function. ***In a function, no two ordered pairs can have the same first component and different second components.***

Different y-values

$$H = \{(-4, 1), (-2, 1), (-2, 0)\} \quad \text{Not a function}$$

Same x-value

Now Try Exercises 11 and 13.



F is a function.

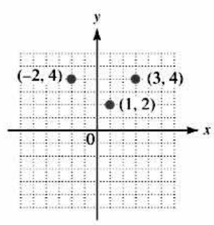
H is not a function.

Figure 19

Relations and functions can also be expressed as a correspondence or *mapping* from one set to another, as shown in **Figure 19** for function F and relation H from **Example 1.** The arrow from 1 to 2 indicates that the ordered pair $(1, 2)$ belongs to F—each first component is paired with exactly one second component. In the mapping for relation H, which is not a function, the first component -2 is paired with two different second components, 1 and 0.

Because relations and functions are sets of ordered pairs, we can represent them using tables and graphs. A table and graph for function F are shown in **Figure 20.**

x	y
1	2
-2	4
3	4



Graph of *F*

Figure 20

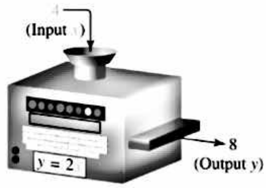
Finally, we can describe a relation or function using a rule that tells how to determine the dependent variable for a specific value of the independent variable. The rule may be given in words: for instance, "the dependent variable is twice the independent variable." Usually the rule is an equation, such as the one below.

$$\text{Dependent variable} \longrightarrow y = 2x \longleftarrow \text{Independent variable}$$

On this particular day, an *input* of pumping 7.870 gallons of gasoline led to an *output* of \$29.58 from the purchaser's wallet. This is an example of a function whose domain consists of numbers of gallons pumped, and whose range consists of amounts from the purchaser's wallet. Dividing the dollar amount by the number of gallons pumped gives the exact price of gasoline that day. Use a calculator to check this. Was this pump fair? (Later we will see that this price is an example of the slope m of a linear function of the form $y = mx$.)

In a function, there is exactly one value of the dependent variable, the second component, for each value of the independent variable, the first component.

NOTE Another way to think of a function relationship is to think of the independent variable as an **input** and of the dependent variable as an **output.** This is illustrated by the **input-output (function) machine** for the function $y = 2x$.



Function machine

Domain and Range We now consider two important concepts concerning relations.

Domain and Range

For every relation consisting of a set of ordered pairs (x, y), there are two important sets of elements.

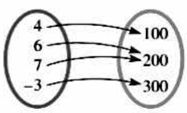
- The set of all values of the independent variable (x) is the **domain.**
- The set of all values of the dependent variable (y) is the **range.**

EXAMPLE 2 Finding Domains and Ranges of Relations

Give the domain and range of each relation. Tell whether the relation defines a function.

(a) $\{(3, -1), (4, 2), (4, 5), (6, 8)\}$

(b)



(c)

x	y
-5	2
0	2
5	2

SOLUTION

(a) The domain is the set of x-values, $\{3, 4, 6\}$. The range is the set of y-values, $\{-1, 2, 5, 8\}$. This relation is not a function because the same x-value, 4, is paired with two different y-values, 2 and 5.

(b) The domain is $\{4, 6, 7, -3\}$ and the range is $\{100, 200, 300\}$. This mapping defines a function. Each x-value corresponds to exactly one y-value.

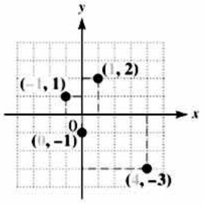
(c) This relation is a set of ordered pairs, so the domain is the set of x-values $\{-5, 0, 5\}$ and the range is the set of y-values $\{2\}$. The table defines a function because each different x-value corresponds to exactly one y-value (even though it is the same y-value).

✔ **Now Try Exercises 19, 21, and 23.**

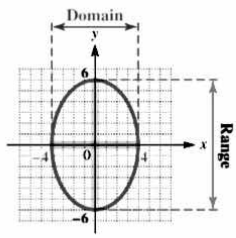
EXAMPLE 3 Finding Domains and Ranges from Graphs

Give the domain and range of each relation.

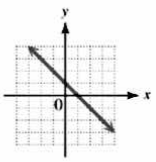
(a)



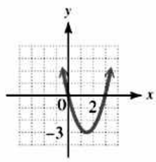
(b)



(c)



(d)



SOLUTION

(a) The domain is the set of x-values, $\{-1, 0, 1, 4\}$. The range is the set of y-values, $\{-3, -1, 1, 2\}$.

(b) The x-values of the points on the graph include all numbers between -4 and 4, inclusive. The y-values include all numbers between -6 and 6, inclusive.

The domain is $[-4, 4]$.

The range is $[-6, 6]$.

Use interval notation.

(c) The arrowheads indicate that the line extends indefinitely left and right, as well as up and down. Therefore, both the domain and the range include all real numbers, which is written $(-\infty, \infty)$.

(d) The arrowheads indicate that the graph extends indefinitely left and right, as well as upward. The domain is $(-\infty, \infty)$. Because there is a least y-value, -3, the range includes all numbers greater than or equal to -3, written $[-3, \infty)$.

Now Try Exercises 27 and 29.

Relations are often defined by equations, such as $y = 2x + 3$ and $y^2 = x$, so we must sometimes determine the domain of a relation from its equation. In this book, we assume the following agreement on the domain of a relation.

Agreement on Domain

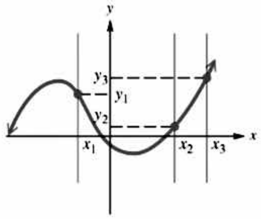
Unless specified otherwise, the domain of a relation is assumed to be all real numbers that produce real numbers when substituted for the independent variable.

To illustrate this agreement, because any real number can be used as a replacement for x in $y = 2x + 3$, the domain of this function is the set of all real numbers.

As another example, the function $y = \frac{1}{x}$ has the set of all real numbers *except* 0 as domain because y is undefined if $x = 0$.

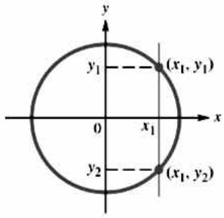
In general, the domain of a function defined by an algebraic expression is the set of all real numbers, except those numbers that lead to division by 0 or to an even root of a negative number.

(There are also exceptions for logarithmic and trigonometric functions. They are covered in further treatment of precalculus mathematics.)



This is the graph of a function. Each x-value corresponds to only one y-value.

(a)



This is not the graph of a function. The same x-value corresponds to two different y-values.

(b)

Figure 21

Determining Whether Relations Are Functions Because each value of x leads to only one value of y in a function, any vertical line must intersect the graph in at most one point. This is the **vertical line test** for a function.

Vertical Line Test

If every vertical line intersects the graph of a relation in no more than one point, then the relation is a function.

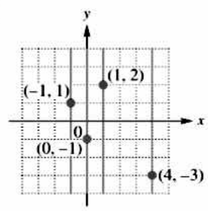
The graph in **Figure 21(a)** represents a function because each vertical line intersects the graph in no more than one point. The graph in **Figure 21(b)** is not the graph of a function because there exists a vertical line that intersects the graph in more than one point.

EXAMPLE 4 Using the Vertical Line Test

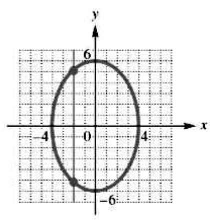
Use the vertical line test to determine whether each relation graphed in **Example 3** is a function.

SOLUTION We repeat each graph from **Example 3,** this time with vertical lines drawn through the graphs.

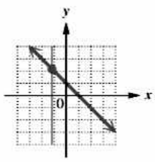
(a)



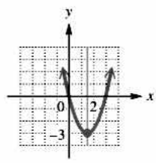
(b)



(c)



(d)



- The graphs of the relations in parts (a), (c), and (d) pass the vertical line test because every vertical line intersects each graph no more than once. Thus, these graphs represent functions.
- The graph of the relation in part (b) fails the vertical line test because the same x-value corresponds to two different y-values. Therefore, it is not the graph of a function.

✔ **Now Try Exercises 27 and 29.**

The vertical line test is a simple method for identifying a function defined by a graph. Deciding whether a relation defined by an equation or an inequality is a function, as well as determining the domain and range, is more difficult. The next example gives some hints that may help.

EXAMPLE 5 Identifying Functions, Domains, and Ranges

Decide whether each relation defines y as a function of x, and give the domain and range.

(a) $y = x + 4$ **(b)** $y = \sqrt{2x - 1}$ **(c)** $y^2 = x$

(d) $y \leq x - 1$ **(e)** $y = \dfrac{5}{x - 1}$

SOLUTION

(a) In the defining equation (or rule), $y = x + 4$, y is always found by adding 4 to x. Thus, each value of x corresponds to just one value of y, and the relation defines a function. The variable x can represent any real number, so the domain is

$$\{x \mid x \text{ is a real number}\}, \quad \text{or} \quad (-\infty, \infty).$$

Because y is always 4 more than x, y also may be any real number, and so the range is $(-\infty, \infty)$.

(b) For any choice of x in the domain of $y = \sqrt{2x - 1}$, there is exactly one corresponding value for y (the radical is a nonnegative number), so this equation defines a function. The equation involves a square root, so the quantity under the radical sign cannot be negative.

$$2x - 1 \geq 0 \quad \text{Solve the inequality.}$$
$$2x \geq 1 \quad \text{Add 1.}$$
$$x \geq \frac{1}{2} \quad \text{Divide by 2.}$$

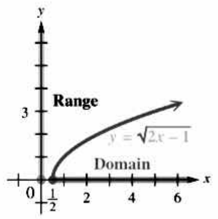


Figure 22

The domain of the function is $\left[\frac{1}{2}, \infty\right)$. Because the radical must represent a nonnegative number, as x takes values greater than or equal to $\frac{1}{2}$, the range is $\{y \mid y \geq 0\}$, or $[0, \infty)$. See **Figure 22.**

(c) The ordered pairs $(16, 4)$ and $(16, -4)$ both satisfy the equation $y^2 = x$. There exists at least one value of x—for example, 16—that corresponds to two values of y, 4 and -4, so this equation does not define a function.

Because x is equal to the square of y, the values of x must always be nonnegative. The domain of the relation is $[0, \infty)$. Any real number can be squared, so the range of the relation is $(-\infty, \infty)$. See **Figure 23.**

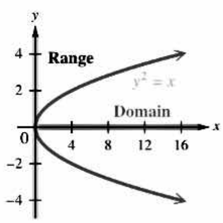


Figure 23

(d) By definition, y is a function of x if every value of x leads to exactly one value of y. Substituting a particular value of x, say 1, into $y \leq x - 1$ corresponds to many values of y. The ordered pairs

$$(1, 0), \quad (1, -1), \quad (1, -2), \quad (1, -3), \quad \text{and so on}$$

all satisfy the inequality, so y is not a function of x here. Any number can be used for x or for y, so the domain and the range of this relation are both the set of real numbers, $(-\infty, \infty)$.

(e) Given any value of x in the domain of

$$y = \frac{5}{x - 1},$$

we find y by subtracting 1 from x, and then dividing the result into 5. This process produces exactly one value of y for each value in the domain, so this equation defines a function.

The domain of $y = \frac{5}{x-1}$ includes all real numbers except those that make the denominator 0. We find these numbers by setting the denominator equal to 0 and solving for x.

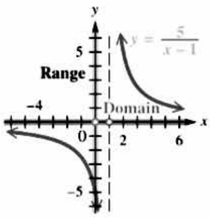


Figure 24

$$x - 1 = 0$$

$$x = 1 \quad \text{Add 1.}$$

Thus, the domain includes all real numbers except 1, written as the interval $(-\infty, 1) \cup (1, \infty)$. Values of y can be positive or negative, but never 0, because a fraction cannot equal 0 unless its numerator is 0. Therefore, the range is the interval $(-\infty, 0) \cup (0, \infty)$, as shown in **Figure 24.**

✔ **Now Try Exercises 37, 39, and 45.**

Variations of the Definition of Function

1. A **function** is a relation in which, for each distinct value of the first component of the ordered pairs, there is exactly one value of the second component.
2. A **function** is a set of ordered pairs in which no first component is repeated.
3. A **function** is a rule or correspondence that assigns exactly one range value to each distinct domain value.

LOOKING AHEAD TO CALCULUS

One of the most important concepts in calculus, that of the **limit of a function,** is defined using function notation:

$$\lim_{x \to a} f(x) = L$$

(read "the limit of $f(x)$ as x approaches a is equal to L") means that the values of $f(x)$ become as close as we wish to L when we choose values of x sufficiently close to a.

Function Notation When a function f is defined with a rule or an equation using x and y for the independent and dependent variables, we say, "*y is a function of x*" to emphasize that *y depends on x*. We use the notation

$$y = f(x),$$

called **function notation,** to express this and read $f(x)$ as **"f of x,"** or **"f at x."** The letter f is the name given to this function.

For example, if $y = 3x - 5$, we can name the function f and write

$$f(x) = 3x - 5.$$

Note that $f(x)$ is just another name for the dependent variable y. For example, if $y = f(x) = 3x - 5$ and $x = 2$, then we find y, or $f(2)$, by replacing x with 2.

$$f(2) = 3 \cdot 2 - 5 \quad \text{Let } x = 2.$$

$$f(2) = 1 \quad \text{Multiply, and then subtract.}$$

The statement "In the function f, if $x = 2$, then $y = 1$" represents the ordered pair $(2, 1)$ and is abbreviated with function notation as follows.

$$f(2) = 1$$

The symbol $f(2)$ is read "f of 2" or "f at 2."

Function notation can be illustrated as follows.

Name of the function

Defining expression

$$y = f(x) = \overbrace{3x - 5}$$

Value of the function　　Name of the independent variable

CAUTION ***The symbol $f(x)$ does not indicate "f times x,"*** but represents the y-value associated with the indicated x-value. As just shown, $f(2)$ is the y-value that corresponds to the x-value 2.

EXAMPLE 6 Using Function Notation

Let $f(x) = -x^2 + 5x - 3$ and $g(x) = 2x + 3$. Find each of the following.

(a) $f(2)$　　**(b)** $f(q)$　　**(c)** $g(a + 1)$

SOLUTION

(a) $f(x) = -x^2 + 5x - 3$

$f(2) = -2^2 + 5 \cdot 2 - 3$　Replace x with 2.

$f(2) = -4 + 10 - 3$　Apply the exponent and multiply.

$f(2) = 3$　Add and subtract.

Thus, $f(2) = 3$, and the ordered pair $(2, 3)$ belongs to f.

(b) $f(x) = -x^2 + 5x - 3$

$f(q) = -q^2 + 5q - 3$　Replace x with q.

(c) $g(x) = 2x + 3$

$g(a + 1) = 2(a + 1) + 3$　Replace x with $a + 1$.

$g(a + 1) = 2a + 2 + 3$　Distributive property

$g(a + 1) = 2a + 5$　Add.

The replacement of one variable with another variable or expression, as in parts (b) and (c), is important in later courses.

✔ Now Try Exercises 51, 59, and 65.

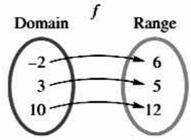
Functions can be evaluated in a variety of ways, as shown in **Example 7.**

EXAMPLE 7 Using Function Notation

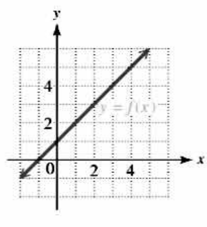
For each function, find $f(3)$.

(a) $f(x) = 3x - 7$　　**(b)** $f = \{(-3, 5), (0, 3), (3, 1), (6, -1)\}$

(c)



(d)



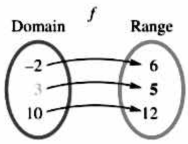


Figure 25

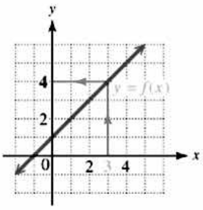


Figure 26

SOLUTION

(a)
$$f(x) = 3x - 7$$
$$f(3) = 3(3) - 7 \quad \text{Replace } x \text{ with } 3.$$
$$f(3) = 2 \quad \text{Simplify.}$$

$f(3) = 2$ indicates that the ordered pair $(3, 2)$ belongs to f.

(b) For $f = \{(-3, 5), (0, 3), (3, 1), (6, -1)\}$, we want $f(3)$, the y-value of the ordered pair where $x = 3$. As indicated by the ordered pair $(3, 1)$, when $x = 3$, $y = 1$, so $f(3) = 1$.

(c) In the mapping, repeated in **Figure 25,** the domain element 3 is paired with 5 in the range, so $f(3) = 5$.

(d) To evaluate $f(3)$ using the graph, find 3 on the x-axis. See **Figure 26.** Then move up until the graph of f is reached. Moving horizontally to the y-axis gives 4 for the corresponding y-value. Thus, $f(3) = 4$.

✔ **Now Try Exercises 67, 69, and 71.**

If a function f is defined by an equation with x and y (and not with function notation), use the following steps to find $f(x)$.

Finding an Expression for $f(x)$

Consider an equation involving x and y. Assume that y can be expressed as a function f of x. To find an expression for $f(x)$, use the following steps.

Step 1 Solve the equation for y.

Step 2 Replace y with $f(x)$.

EXAMPLE 8 Writing Equations Using Function Notation

Assume that y is a function f of x. Rewrite each equation using function notation. Then find $f(-2)$ and $f(p)$.

(a) $y = x^2 + 1$ **(b)** $x - 4y = 5$

SOLUTION

(a) ***Step 1*** $y = x^2 + 1$ This equation is already solved for y.

Step 2 $f(x) = x^2 + 1$ Let $y = f(x)$.

Now find $f(-2)$ and $f(p)$.

$$f(-2) = (-2)^2 + 1 \quad \text{Let } x = -2.$$
$$f(-2) = 4 + 1$$
$$f(-2) = 5$$

$$f(p) = p^2 + 1 \quad \text{Let } x = p.$$

(b) ***Step 1***
$$x - 4y = 5 \quad \text{Given equation.}$$
$$-4y = -x + 5 \quad \text{Add } -x.$$
$$y = \frac{x - 5}{4} \quad \text{Multiply by } -1. \text{ Divide by } 4.$$

Step 2
$$f(x) = \frac{1}{4}x - \frac{5}{4} \quad \text{Let } y = f(x); \ \frac{a - b}{c} = \frac{a}{c} - \frac{b}{c}.$$

Now find $f(-2)$ and $f(p)$.

$$f(x) = \frac{1}{4}x - \frac{5}{4}$$

$$f(-2) = \frac{1}{4}(-2) - \frac{5}{4} \quad \text{Let } x = -2.$$

$$f(-2) = -\frac{7}{4}$$

$$f(p) = \frac{1}{4}p - \frac{5}{4} \quad \text{Let } x = p.$$

Now Try Exercises 77 and 81.

Increasing, Decreasing, and Constant Functions Informally speaking, a function *increases* over an open interval of its domain if its graph rises from left to right on the interval. It *decreases* over an open interval of its domain if its graph falls from left to right on the interval. It is *constant* over an open interval of its domain if its graph is horizontal on the interval.

For example, consider **Figure 27.**

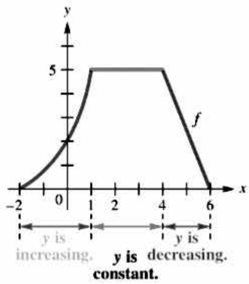


Figure 27

- The function increases over the open interval $(-2, 1)$ because the y-values continue to get larger for x-values in that interval.
- The function is constant over the open interval $(1, 4)$ because the y-values are always 5 for all x-values there.
- The function decreases over the open interval $(4, 6)$ because in that interval the y-values continuously get smaller.

The intervals refer to the x-values where the y-values either increase, decrease, or are constant.

The formal definitions of these concepts follow.

Increasing, Decreasing, and Constant Functions

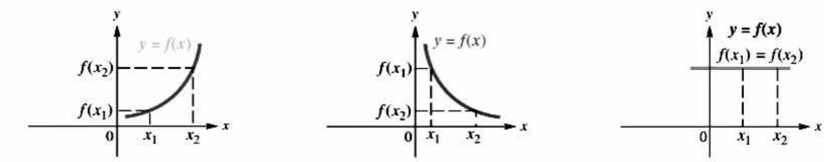
Suppose that a function f is defined over an *open* interval I and x_1 and x_2 are in I.

(a) f **increases** over I if, whenever $x_1 < x_2$, $f(x_1) < f(x_2)$.

(b) f **decreases** over I if, whenever $x_1 < x_2$, $f(x_1) > f(x_2)$.

(c) f is **constant** over I if, for every x_1 and x_2, $f(x_1) = f(x_2)$.

Figure 28 illustrates these ideas.



Whenever $x_1 < x_2$, and $f(x_1) < f(x_2)$, f is increasing.
(a)

Whenever $x_1 < x_2$, and $f(x_1) > f(x_2)$, f is decreasing.
(b)

For every x_1 and x_2, if $f(x_1) = f(x_2)$, then f is constant.
(c)

Figure 28

NOTE To decide whether a function is increasing, decreasing, or constant over an interval, ask yourself, ***"What does y do as x goes from left to right?"*** Our definition of *increasing, decreasing,* and *constant* function behavior applies to open intervals of the domain, not to individual points.

EXAMPLE 9 Determining Increasing, Decreasing, and Constant Intervals

Figure 29 shows the graph of a function. Determine the largest open intervals of the domain over which the function is increasing, decreasing, or constant.

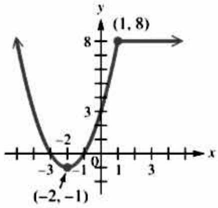


Figure 29

SOLUTION We observe the domain and ask, "*What is happening to the y-values as the x-values are getting larger?*" Moving from left to right on the graph, we see the following:

- On the open interval $(-\infty, -2)$, the y-values are *decreasing*.
- On the open interval $(-2, 1)$, the y-values are *increasing*.
- On the open interval $(1, \infty)$, the y-values are *constant* (and equal to 8).

Therefore, the function is decreasing on $(-\infty, -2)$, increasing on $(-2, 1)$, and constant on $(1, \infty)$.

✔ **Now Try Exercise 91.**

EXAMPLE 10 Interpreting a Graph

Figure 30 shows the relationship between the number of gallons, $g(t)$, of water in a small swimming pool and time in hours, t. By looking at this graph of the function, we can answer questions about the water level in the pool at various times. For example, at time 0 the pool is empty. The water level then increases, stays constant for a while, decreases, and then becomes constant again.

Use the graph to respond to the following.

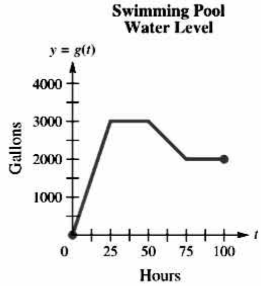


Figure 30

(a) What is the maximum number of gallons of water in the pool? When is the maximum water level first reached?

(b) For how long is the water level increasing? decreasing? constant?

(c) How many gallons of water are in the pool after 90 hr?

(d) Describe a series of events that could account for the water level changes shown in the graph.