CONCEPT PREVIEW *Match each equation in Column I with its graph in Column II.*

I

5. $(x-3)^2+(y-2)^2=25$

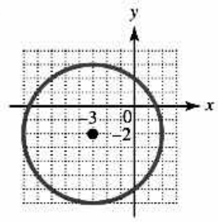
6. $(x-3)^2+(y+2)^2=25$

7. $(x+3)^2+(y-2)^2=25$

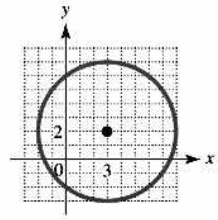
8. $(x+3)^2+(y+2)^2=25$

II

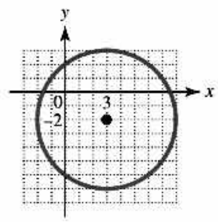
A.



B.



C.



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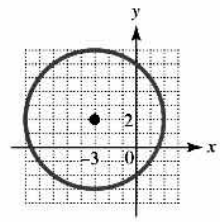


CONCEPT PREVIEW *Answer each question.*

9. How many points lie on the graph of $x^2+y^2=0$?

10. How many points lie on the graph of $x^2+y^2=-100$?

In the following exercises, ***(a)*** *find the center-radius form of the equation of each circle described, and* ***(b)*** *graph it. See* ***Examples 1 and 2.***

11. center $(0, 0)$, radius 6

12. center $(0, 0)$, radius 9

13. center $(2, 0)$, radius 6

14. center $(3, 0)$, radius 3

15. center $(0, 4)$, radius 4

16. center $(0, -3)$, radius 7

17. center $(-2, 5)$, radius 4

18. center $(4, 3)$, radius 5

19. center $(5, -4)$, radius 7

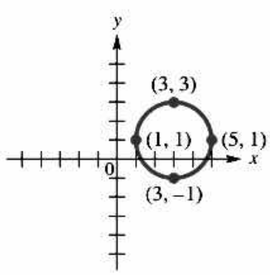
20. center $(-3, -2)$, radius 6

21. center $\left(\sqrt{2}, \sqrt{2}\right)$, radius $\sqrt{2}$

22. center $\left(-\sqrt{3}, -\sqrt{3}\right)$, radius $\sqrt{3}$

Connecting Graphs with Equations *Use each graph to determine an equation of the circle in* ***(a)*** *center-radius form and* ***(b)*** *general form.*

23.



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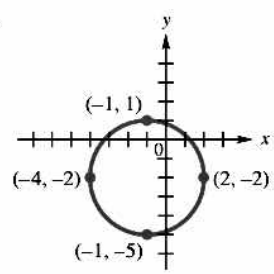
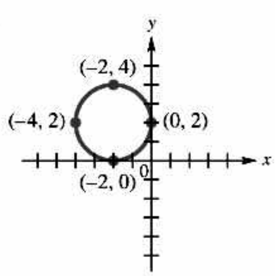
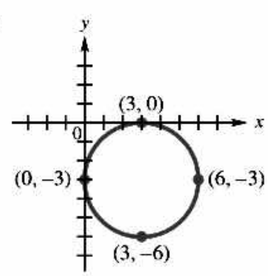


25.



26.



Decide whether or not each equation has a circle as its graph. If it does, give the center and radius. If it does not, describe the graph. ***See Examples 3–5.***

27. $x^2 + y^2 + 6x + 8y + 9 = 0$

28. $x^2 + y^2 + 8x - 6y + 16 = 0$

29. $x^2 + y^2 - 4x + 12y = -4$

30. $x^2 + y^2 - 12x + 10y = -25$

31. $4x^2 + 4y^2 + 4x - 16y - 19 = 0$

32. $9x^2 + 9y^2 + 12x - 18y - 23 = 0$

33. $x^2 + y^2 + 2x - 6y + 14 = 0$

34. $x^2 + y^2 + 4x - 8y + 32 = 0$

35. $x^2 + y^2 - 6x - 6y + 18 = 0$

36. $x^2 + y^2 + 4x + 4y + 8 = 0$

37. $9x^2 + 9y^2 - 6x + 6y - 23 = 0$

38. $4x^2 + 4y^2 + 4x - 4y - 7 = 0$

Epicenter of an Earthquake *Solve each problem. To visualize the situation, use graph paper and a compass to carefully graph each circle.* ***See Example 6.***

39. Suppose that receiving stations X, Y, and Z are located on a coordinate plane at the points

$$(7, 4), \quad (-9, -4), \quad \text{and} \quad (-3, 9),$$

respectively. The epicenter of an earthquake is determined to be 5 units from X, 13 units from Y, and 10 units from Z. Where on the coordinate plane is the epicenter located?

40. Suppose that receiving stations P, Q, and R are located on a coordinate plane at the points

$$(3, 1), \quad (5, -4), \quad \text{and} \quad (-1, 4),$$

respectively. The epicenter of an earthquake is determined to be $\sqrt{5}$ units from P, 6 units from Q, and $2\sqrt{10}$ units from R. Where on the coordinate plane is the epicenter located?

41. The locations of three receiving stations and the distances to the epicenter of an earthquake are contained in the following three equations:

$$(x - 2)^2 + (y - 1)^2 = 25, \quad (x + 2)^2 + (y - 2)^2 = 16,$$
$$\text{and} \quad (x - 1)^2 + (y + 2)^2 = 9.$$

Determine the location of the epicenter.

42. The locations of three receiving stations and the distances to the epicenter of an earthquake are contained in the following three equations:

$$(x - 2)^2 + (y - 4)^2 = 25, \quad (x - 1)^2 + (y + 3)^2 = 25,$$
$$\text{and} \quad (x + 3)^2 + (y + 6)^2 = 100.$$

Determine the location of the epicenter.

Concept Check *Work each of the following.*

43. Find the center-radius form of the equation of a circle with center $(3, 2)$ and tangent to the x-axis. (*Hint:* A line **tangent** to a circle touches it at exactly one point.)

44. Find the equation of a circle with center at $(-4, 3)$, passing through the point $(5, 8)$. Write it in center-radius form.

45. Find all points (x, y) with $x = y$ that are 4 units from $(1, 3)$.

46. Find all points satisfying $x + y = 0$ that are 8 units from $(-2, 3)$.

47. Find the coordinates of all points whose distance from $(1, 0)$ is $\sqrt{10}$ and whose distance from $(5, 4)$ is $\sqrt{10}$.

2.2 Circles

- Center-Radius Form
- General Form
- An Application

Center-Radius Form By definition, a **circle** is the set of all points in a plane that lie a given distance from a given point. The given distance is the **radius** of the circle, and the given point is the **center.**

We can find the equation of a circle from its definition using the distance formula. Suppose that the point (h, k) is the center and the circle has radius r, where $r > 0$. Let (x, y) represent any point on the circle. See **Figure 13.**

LOOKING AHEAD TO CALCULUS
The circle $x^2 + y^2 = 1$ is called the **unit circle.** It is important in interpreting the *trigonometric* or *circular* functions that appear in the study of calculus.

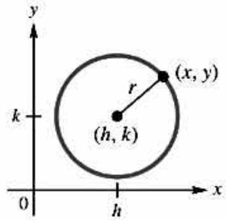


Figure 13

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d \qquad \text{Distance formula}$$

$$\sqrt{(x - h)^2 + (y - k)^2} = r \qquad (x_1, y_1) = (h, k),\ (x_2, y_2) = (x, y), \text{ and } d = r$$

$$(x - h)^2 + (y - k)^2 = r^2 \qquad \text{Square each side.}$$

Center-Radius Form of the Equation of a Circle

A circle with center (h, k) and radius r has equation

$$(x - h)^2 + (y - k)^2 = r^2,$$

which is the **center-radius form** of the equation of the circle. As a special case, a circle with center $(0, 0)$ and radius r has the following equation.

$$x^2 + y^2 = r^2$$

EXAMPLE 1 Finding the Center-Radius Form

Find the center-radius form of the equation of each circle described.

(a) center $(-3, 4)$, radius 6 **(b)** center $(0, 0)$, radius 3

SOLUTION

(a)

$$(x - h)^2 + (y - k)^2 = r^2 \qquad \text{Center-radius form}$$

$$[x - (-3)]^2 + (y - 4)^2 = 6^2 \qquad \text{Substitute. Let } (h, k) = (-3, 4) \text{ and } r = 6.$$

Be careful with signs here.

$$(x + 3)^2 + (y - 4)^2 = 36 \qquad \text{Simplify.}$$

(b) The center is the origin and $r = 3$.

$$x^2 + y^2 = r^2 \qquad \text{Special case of the center-radius form}$$

$$x^2 + y^2 = 3^2 \qquad \text{Let } r = 3.$$

$$x^2 + y^2 = 9 \qquad \text{Apply the exponent.}$$

✔ **Now Try Exercises 11(a) and 17(a).**

EXAMPLE 2 Graphing Circles

Graph each circle discussed in **Example 1.**

(a) $(x + 3)^2 + (y - 4)^2 = 36$ **(b)** $x^2 + y^2 = 9$

SOLUTION

(a) Writing the given equation in center-radius form

$$[x - (-3)]^2 + (y - 4)^2 = 6^2$$

gives $(-3, 4)$ as the center and 6 as the radius. See **Figure 14.**

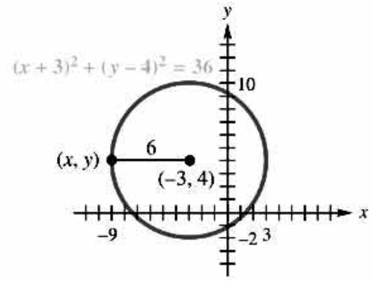


Figure 14

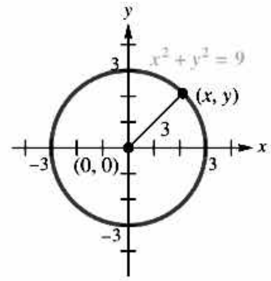
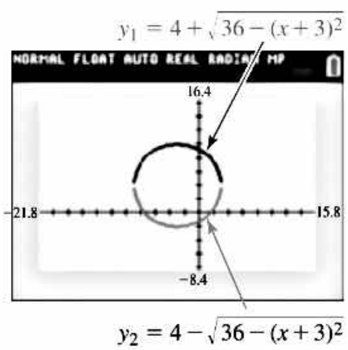


Figure 15

(b) The graph with center $(0, 0)$ and radius 3 is shown in **Figure 15.**

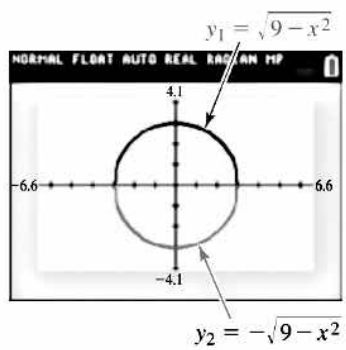
✔ **Now Try Exercises 11(b) and 17(b).**

The circles graphed in **Figures 14 and 15** of **Example 2** can be generated on a graphing calculator by first solving for y and then entering two expressions for y_1 and y_2. See **Figures 16 and 17.** In both cases, the plot of y_1 yields the top half of the circle, and that of y_2 yields the bottom half. It is necessary to use a **square viewing window** to avoid distortion when graphing circles.



The graph of this circle has equation $(x + 3)^2 + (y - 4)^2 = 36.$

Figure 16



The graph of this circle has equation $x^2 + y^2 = 9.$

Figure 17

General Form Consider the center-radius form of the equation of a circle, and rewrite it so that the binomials are expanded and the right side equals 0.

Don't forget the middle term when squaring each binomial.

$$(x - h)^2 + (y - k)^2 = r^2 \quad \text{Center-radius form}$$

$$x^2 - 2xh + h^2 + y^2 - 2yk + k^2 - r^2 = 0 \quad \text{Square each binomial, and subtract } r^2.$$

$$x^2 + y^2 + \underbrace{(-2h)}_{D}x + \underbrace{(-2k)}_{E}y + \underbrace{(h^2 + k^2 - r^2)}_{F} = 0 \quad \text{Properties of real numbers}$$

If $r > 0$, then the graph of this equation is a circle with center (h, k) and radius r.

This form is the **general form of the equation of a circle.**

General Form of the Equation of a Circle

For some real numbers D, E, and F, the equation

$$\mathbf{x^2 + y^2 + Dx + Ey + F = 0}$$

can have a graph that is a circle or a point, or is nonexistent.

Starting with an equation in this general form, we can complete the square to get an equation of the form

$$(x-h)^2 + (y-k)^2 = c, \quad \text{for some number } c.$$

There are three possibilities for the graph based on the value of c.

1. If $c > 0$, then $r^2 = c$, and the graph of the equation is a circle with radius $\sqrt{c}$.
2. If $c = 0$, then the graph of the equation is the single point (h, k).
3. If $c < 0$, then no points satisfy the equation, and the graph is nonexistent.

The next example illustrates the procedure for finding the center and radius.

EXAMPLE 3 Finding the Center and Radius by Completing the Square

Show that $x^2 - 6x + y^2 + 10y + 18 = 0$ has a circle as its graph. Find the center and radius.

SOLUTION We complete the square twice, once for x and once for y. Begin by subtracting 18 from each side.

$$x^2 - 6x + y^2 + 10y + 18 = 0$$

$$(x^2 - 6x \qquad) + (y^2 + 10y \qquad) = -18$$

Think: $\left[\frac{1}{2}(-6)\right]^2 = (-3)^2 = 9$ and $\left[\frac{1}{2}(10)\right]^2 = 5^2 = 25$

Add 9 and 25 on the left to complete the two squares, and to compensate, add 9 and 25 on the right.

$$(x^2 - 6x + 9) + (y^2 + 10y + 25) = -18 + 9 + 25 \qquad \text{Complete the square.}$$

Add 9 and 25 on *both* sides.

$$(x-3)^2 + (y+5)^2 = 16 \qquad \text{Factor. Add on the right.}$$

$$(x-3)^2 + [y-(-5)]^2 = 4^2 \qquad \text{Center-radius form}$$

Because $4^2 = 16$ and $16 > 0$, the equation represents a circle with center $(3, -5)$ and radius 4.

✔ **Now Try Exercise 27.**

EXAMPLE 4 Finding the Center and Radius by Completing the Square

Show that $2x^2 + 2y^2 - 6x + 10y = 1$ has a circle as its graph. Find the center and radius.

SOLUTION To complete the square, the coefficient of the x^2-term and that of the y^2-term must be 1. In this case they are both 2, so begin by dividing each side by 2.

$$2x^2 + 2y^2 - 6x + 10y = 1$$

$$x^2 + y^2 - 3x + 5y = \frac{1}{2} \qquad \text{Divide by 2.}$$

$$(x^2 - 3x \qquad) + (y^2 + 5y \qquad) = \frac{1}{2} \qquad \text{Rearrange and regroup terms in anticipation of completing the square.}$$

$$\left(x^2 - 3x + \frac{9}{4}\right) + \left(y^2 + 5y + \frac{25}{4}\right) = \frac{1}{2} + \frac{9}{4} + \frac{25}{4} \qquad \text{Complete the square for } \textit{both } x \text{ and } y\text{; } \left[\tfrac{1}{2}(-3)\right]^2 = \tfrac{9}{4} \text{ and } \left[\tfrac{1}{2}(5)\right]^2 = \tfrac{25}{4}.$$

$$\left(x - \frac{3}{2}\right)^2 + \left(y + \frac{5}{2}\right)^2 = 9 \qquad \text{Factor and add.}$$

$$\left(x - \frac{3}{2}\right)^2 + \left[y - \left(-\frac{5}{2}\right)\right]^2 = 3^2 \qquad \text{Center-radius form}$$

The equation has a circle with center $\left(\frac{3}{2}, -\frac{5}{2}\right)$ and radius 3 as its graph.

✔ **Now Try Exercise 31.**

EXAMPLE 5 Determining Whether a Graph Is a Point or Nonexistent

The graph of the equation $x^2 + 10x + y^2 - 4y + 33 = 0$ either is a point or is nonexistent. Which is it?

SOLUTION

$$x^2 + 10x + y^2 - 4y + 33 = 0$$

$$x^2 + 10x + y^2 - 4y = -33 \qquad \text{Subtract 33.}$$

Think: $\left[\frac{1}{2}(10)\right]^2 = 25$ and $\left[\frac{1}{2}(-4)\right]^2 = 4$ Prepare to complete the square for both x and y.

$$(x^2 + 10x + 25) + (y^2 - 4y + 4) = -33 + 25 + 4 \qquad \text{Complete the square.}$$

$$(x + 5)^2 + (y - 2)^2 = -4 \qquad \text{Factor and add.}$$

Because $-4 < 0$, there are *no* ordered pairs (x, y), with x and y both real numbers, satisfying the equation. The graph of the given equation is nonexistent—it contains no points. (If the constant on the right side were 0, the graph would consist of the single point $(-5, 2)$.)

✔ **Now Try Exercise 33.**

An Application Seismologists can locate the epicenter of an earthquake by determining the intersection of three circles. The radii of these circles represent the distances from the epicenter to each of three receiving stations. The centers of the circles represent the receiving stations.