2.1 Exercises

CONCEPT PREVIEW *Fill in the blank to correctly complete each sentence.*

1. The point $(-1, 3)$ lies in quadrant ______ in the rectangular coordinate system.
2. The point $(4, \_\_)$ lies on the graph of the equation $y = 3x - 6$.
3. Any point that lies on the x-axis has y-coordinate equal to ______.
4. The y-intercept of the graph of $y = -2x + 6$ is ______.
5. The x-intercept of the graph of $2x + 5y = 10$ is ______.
6. The distance from the origin to the point $(-3, 4)$ is ______.

CONCEPT PREVIEW *Determine whether each statement is* true *or* false. *If false, explain why.*

7. The graph of $y = x^2 + 2$ has no x-intercepts.
8. The graph of $y = x^2 - 2$ has two x-intercepts.
9. The midpoint of the segment joining $(0, 0)$ and $(4, 4)$ is 2.
10. The distance between the points $(0, 0)$ and $(4, 4)$ is 4.

Give three ordered pairs from each table. **See Example 1.**

11.

x	y
2	-5
-1	7
3	-9
5	-17
6	-21

12.

x	y
3	3
-5	-21
8	18
4	6
0	-6

13. ***Percent of High School Students Who Smoke***

Year	Percent
1999	35
2001	29
2003	22
2005	23
2007	20
2009	20

Source: Centers for Disease Control and Prevention.

14. ***Number of U.S. Viewers of the Super Bowl***

Year	Viewers (millions)
2002	86.8
2004	89.8
2006	90.7
2008	97.4
2010	106.5
2012	111.4
2014	111.5

Source: www.tvbythenumbers.com

For the points P and Q, find **(a)** *the distance $d(P, Q)$ and* **(b)** *the coordinates of the midpoint M of line segment PQ.* **See Examples 2 and 5(a).**

15. $P(-5, -6), Q(7, -1)$
16. $P(-4, 3), Q(2, -5)$
17. $P(8, 2), Q(3, 5)$
18. $P(-8, 4), Q(3, -5)$
19. $P(-6, -5), Q(6, 10)$
20. $P(6, -2), Q(4, 6)$
21. $P(3\sqrt{2}, 4\sqrt{5}), Q(\sqrt{2}, -\sqrt{5})$
22. $P(-\sqrt{7}, 8\sqrt{3}), Q(5\sqrt{7}, -\sqrt{3})$

Determine whether the three points are the vertices of a right triangle. ***See Example 3.***

23. $(-6, -4), (0, -2), (-10, 8)$

24. $(-2, -8), (0, -4), (-4, -7)$

25. $(-4, 1), (1, 4), (-6, -1)$

26. $(-2, -5), (1, 7), (3, 15)$

27. $(-4, 3), (2, 5), (-1, -6)$

28. $(-7, 4), (6, -2), (0, -15)$

Determine whether the three points are collinear. ***See Example 4.***

29. $(0, -7), (-3, 5), (2, -15)$

30. $(-1, 4), (-2, -1), (1, 14)$

31. $(0, 9), (-3, -7), (2, 19)$

32. $(-1, -3), (-5, 12), (1, -11)$

33. $(-7, 4), (6, -2), (-1, 1)$

34. $(-4, 3), (2, 5), (-1, 4)$

Find the coordinates of the other endpoint of each line segment, given its midpoint and one endpoint. ***See Example 5(b).***

35. midpoint $(5, 8)$, endpoint $(13, 10)$

36. midpoint $(-7, 6)$, endpoint $(-9, 9)$

37. midpoint $(12, 6)$, endpoint $(19, 16)$

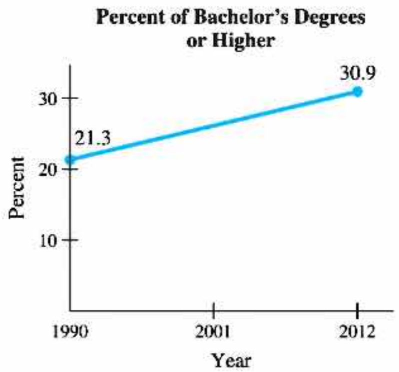
38. midpoint $(-9, 8)$, endpoint $(-16, 9)$

39. midpoint (a, b), endpoint (p, q)

40. midpoint $(6a, 6b)$, endpoint $(3a, 5b)$

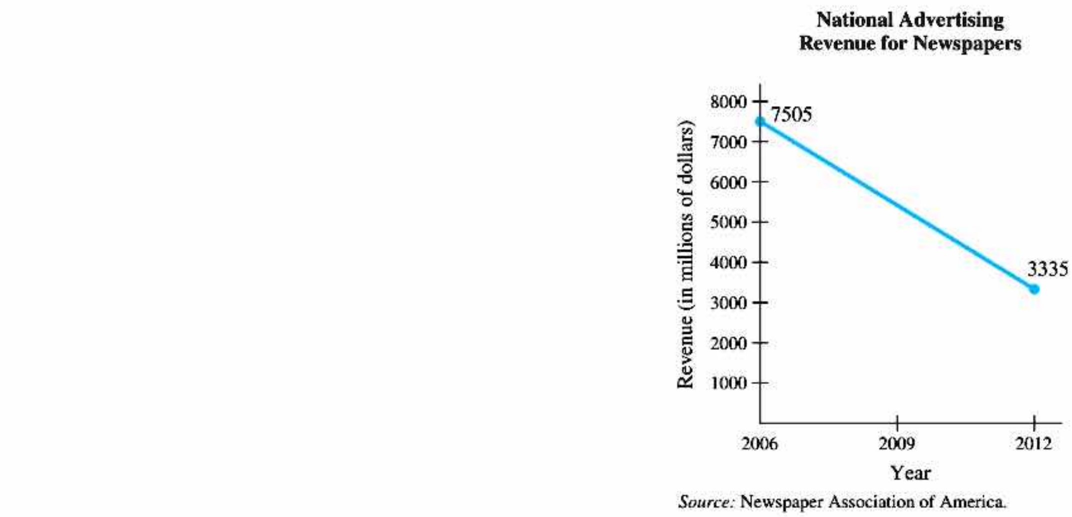
Solve each problem. ***See Example 6.***

41. ***Bachelor's Degree Attainment*** The graph shows a straight line that approximates the percentage of Americans 25 years and older who had earned bachelor's degrees or higher for the years 1990–2012. Use the midpoint formula and the two given points to estimate the percent in 2001. Compare the answer with the actual percent of 26.2.



Source: U.S. Census Bureau.

42. ***Newspaper Advertising Revenue*** The graph shows a straight line that approximates national advertising revenue, in millions of dollars, for newspapers in the United States for the years 2006–2012. Use the midpoint formula and the two given points to estimate revenue in 2009. Compare the answer with the actual figure of 4424 million dollars.



Source: Newspaper Association of America.

2.1 Rectangular Coordinates and Graphs

- Ordered Pairs
- The Rectangular Coordinate System
- The Distance Formula
- The Midpoint Formula
- Equations in Two Variables

Category	Amount Spent
food	\$ 8506
housing	\$21,374
transportation	\$12,153
health care	\$ 4917
apparel and services	\$ 2076
entertainment	\$ 3240

Source: U.S. Bureau of Labor Statistics.

Ordered Pairs

The idea of pairing one quantity with another is often encountered in everyday life.

- A numerical score in a mathematics course is paired with a corresponding letter grade.
- The number of gallons of gasoline pumped into a tank is paired with the amount of money needed to purchase it.
- Expense categories are paired with dollars spent by the average American household in 2013. (See the table in the margin.)

Pairs of related quantities, such as a 96 determining a grade of A, 3 gallons of gasoline costing \$10.50, and 2013 spending on food of \$8506, can be expressed as *ordered pairs:* (96, A), (3, \$10.50), (food, \$8506). An **ordered pair** consists of two components, written inside parentheses.

EXAMPLE 1 Writing Ordered Pairs

Use the table to write ordered pairs to express the relationship between each category and the amount spent on it.

(a) housing **(b)** entertainment

SOLUTION

(a) Use the data in the second row: (housing, \$21,374).

(b) Use the data in the last row: (entertainment, \$3240).

Now Try Exercise 13.

In mathematics, we are most often interested in ordered pairs whose components are numbers. The ordered pairs (a, b) and (c, d) are equal provided that $a = c$ *and* $b = d$.

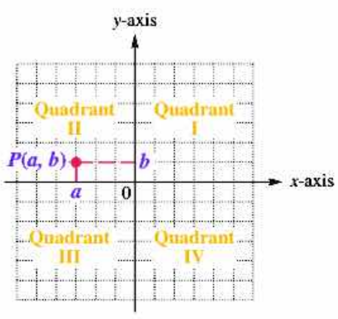
NOTE Notation such as $(2, 4)$ is used to show an interval on a number line, and the same notation is used to indicate an ordered pair of numbers. The intended use is usually clear from the context of the discussion.

The Rectangular Coordinate System

Each real number corresponds to a point on a number line. This idea is extended to ordered pairs of real numbers by using two perpendicular number lines, one horizontal and one vertical, that intersect at their zero-points. The point of intersection is the **origin.** The horizontal line is the ***x*-axis,** and the vertical line is the ***y*-axis.**

The x-axis and y-axis together make up a **rectangular coordinate system,** or **Cartesian coordinate system** (named for one of its coinventors, René Descartes. The other coinventor was Pierre de Fermat). The plane into which the coordinate system is introduced is the **coordinate plane,** or ***xy*-plane.** See **Figure 1.** The x-axis and y-axis divide the plane into four regions, or **quadrants,** labeled as shown. The points on the x-axis or the y-axis belong to no quadrant.

Each point P in the xy-plane corresponds to a unique ordered pair (a, b) of real numbers. The point P corresponding to the ordered pair (a, b) often is written $P(a, b)$ as in **Figure 1** and referred to as "the point (a, b)." The numbers a and b are the **coordinates** of point P.



Rectangular (Cartesian) Coordinate System

Figure 1

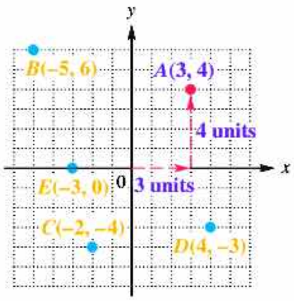


Figure 2

To locate on the xy-plane the point corresponding to the ordered pair $(3, 4)$, for example, start at the origin, move 3 units in the positive x-direction, and then move 4 units in the positive y-direction. See **Figure 2.** Point A corresponds to the ordered pair $(3, 4)$.

The Distance Formula Recall that the distance on a number line between points P and Q with coordinates x_1 and x_2 is

$$d(P, Q) = |x_1 - x_2| = |x_2 - x_1|. \quad \text{Definition of distance}$$

By using the coordinates of their ordered pairs, we can extend this idea to find the distance between any two points in a plane.

Figure 3 shows the points $P(-4, 3)$ and $R(8, -2)$. If we complete a right triangle that has its 90° angle at $Q(8, 3)$ as in the figure, the legs have lengths

$$d(P, Q) = |8 - (-4)| = 12$$

and $$d(Q, R) = |3 - (-2)| = 5.$$

By the Pythagorean theorem, the hypotenuse has length

$$\sqrt{12^2 + 5^2} = \sqrt{144 + 25} = \sqrt{169} = 13.$$

Thus, the distance between $(-4, 3)$ and $(8, -2)$ is 13.

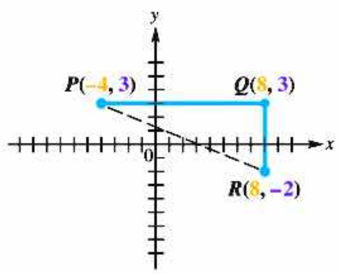


Figure 3

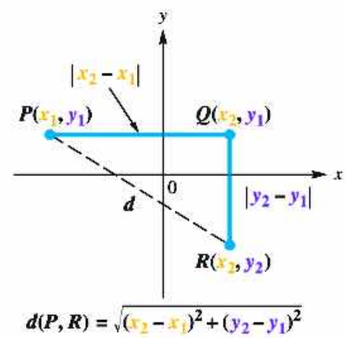


Figure 4

René Descartes (1596–1650)

The initial flash of *analytic geometry* may have come to Descartes as he was watching a fly crawling about on the ceiling near a corner of his room. It struck him that the path of the fly on the ceiling could be described if only one knew the relation connecting the fly's distances from two adjacent walls.

Source: An Introduction to the History of Mathematics by Howard Eves.

To obtain a general formula, let $P(x_1, y_1)$ and $R(x_2, y_2)$ be any two distinct points in a plane, as shown in **Figure 4.** Complete a triangle by locating point Q with coordinates (x_2, y_1). The Pythagorean theorem gives the distance between P and R.

$$d(P, R) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Absolute value bars are not necessary in this formula because, for all real numbers a and b,

$$|a - b|^2 = (a - b)^2.$$

The **distance formula** can be summarized as follows.

Distance Formula

Suppose that $P(x_1, y_1)$ and $R(x_2, y_2)$ are two points in a coordinate plane. The distance between P and R, written $d(P, R)$, is given by the following formula.

$$\mathbf{d(P, R) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

LOOKING AHEAD TO CALCULUS

In analytic geometry and calculus, the distance formula is extended to two points in space. Points in space can be represented by **ordered triples.** The distance between the two points

$$(x_1, y_1, z_1) \quad \text{and} \quad (x_2, y_2, z_2)$$

is given by the following expression.

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The distance formula can be stated in words.

The distance between two points in a coordinate plane is the square root of the sum of the square of the difference between their x-coordinates and the square of the difference between their y-coordinates.

Although our derivation of the distance formula assumed that P and R are not on a horizontal or vertical line, the result is true for any two points.

EXAMPLE 2 Using the Distance Formula

Find the distance between $P(-8, 4)$ and $Q(3, -2)$.

SOLUTION Use the distance formula.

$$\begin{aligned} d(P, Q) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} && \text{Distance formula} \\ &= \sqrt{[3 - (-8)]^2 + (-2 - 4)^2} && x_1 = -8,\ y_1 = 4,\ x_2 = 3,\ y_2 = -2 \\ &= \sqrt{11^2 + (-6)^2} && \text{Be careful when subtracting a negative number.} \\ &= \sqrt{121 + 36} \\ &= \sqrt{157} \end{aligned}$$

Now Try Exercise 15(a).

A statement of the form "If p, then q" is a **conditional statement.** The related statement "If q, then p" is its **converse.** The *converse* of the Pythagorean theorem is also a true statement.

If the sides a, b, and c of a triangle satisfy $a^2 + b^2 = c^2$, then the triangle is a right triangle with legs having lengths a and b and hypotenuse having length c.

EXAMPLE 3 Applying the Distance Formula

Determine whether the points $M(-2, 5)$, $N(12, 3)$, and $Q(10, -11)$ are the vertices of a right triangle.

SOLUTION A triangle with the three given points as vertices, shown in **Figure 5,** is a right triangle if the square of the length of the longest side equals the sum of the squares of the lengths of the other two sides. Use the distance formula to find the length of each side of the triangle.

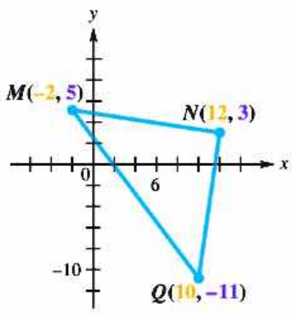


Figure 5

$$d(M, N) = \sqrt{[12 - (-2)]^2 + (3 - 5)^2} = \sqrt{196 + 4} = \sqrt{200}$$

$$d(M, Q) = \sqrt{[10 - (-2)]^2 + (-11 - 5)^2} = \sqrt{144 + 256} = \sqrt{400} = 20$$

$$d(N, Q) = \sqrt{(10 - 12)^2 + (-11 - 3)^2} = \sqrt{4 + 196} = \sqrt{200}$$

The longest side, of length 20 units, is chosen as the hypotenuse. Because

$$\left(\sqrt{200}\right)^2 + \left(\sqrt{200}\right)^2 = 400 = 20^2$$

is true, the triangle is a right triangle with hypotenuse joining M and Q.

Now Try Exercise 23.

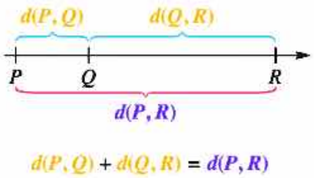


Figure 6

Using a similar procedure, we can tell whether three points are **collinear** (that is, lying on a straight line). See **Figure 6.**

Three points are collinear if the sum of the distances between two pairs of the points is equal to the distance between the remaining pair of points.

EXAMPLE 4 Applying the Distance Formula

Determine whether the points $P(-1, 5)$, $Q(2, -4)$, and $R(4, -10)$ are collinear.

SOLUTION Use the distance formula.

$$d(P, Q) = \sqrt{(-1-2)^2 + [5-(-4)]^2} = \sqrt{9+81} = \sqrt{90} = 3\sqrt{10}$$

$$\sqrt{90} = \sqrt{9 \cdot 10} = 3\sqrt{10}$$

$$d(Q, R) = \sqrt{(2-4)^2 + [-4-(-10)]^2} = \sqrt{4+36} = \sqrt{40} = 2\sqrt{10}$$

$$d(P, R) = \sqrt{(-1-4)^2 + [5-(-10)]^2} = \sqrt{25+225} = \sqrt{250} = 5\sqrt{10}$$

Because $3\sqrt{10} + 2\sqrt{10} = 5\sqrt{10}$ is true, the three points are collinear.

Now Try Exercise 29.

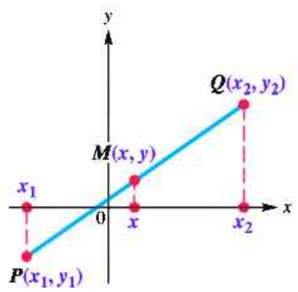


Figure 7

The Midpoint Formula The midpoint of a line segment is equidistant from the endpoints of the segment. The **midpoint formula** is used to find the coordinates of the midpoint of a line segment. To develop the midpoint formula, let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two distinct points in a plane. (Although **Figure 7** shows $x_1 < x_2$, no particular order is required.) Let $M(x, y)$ be the midpoint of the segment joining P and Q. Draw vertical lines from each of the three points to the x-axis, as shown in **Figure 7.**

The ordered pair $M(x, y)$ is the midpoint of the line segment joining P and Q, so the distance between x and x_1 equals the distance between x and x_2.

$$x_2 - x = x - x_1$$

$$x_2 + x_1 = 2x \qquad \text{Add } x \text{ and } x_1 \text{ to each side.}$$

$$x = \frac{x_1 + x_2}{2} \qquad \text{Divide by 2 and rewrite.}$$

Similarly, the y-coordinate is $\frac{y_1 + y_2}{2}$, yielding the following formula.

Midpoint Formula

The coordinates of the midpoint M of the line segment with endpoints $P(x_1, y_1)$ and $Q(x_2, y_2)$ are given by the following.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

That is, the x-coordinate of the midpoint of a line segment is the* average *of the x-coordinates of the segment's endpoints, and the y-coordinate is the* average *of the y-coordinates of the segment's endpoints.