

11. Explain how to determine whether to use a parenthesis or a square bracket when writing the solution set of a linear inequality in interval notation.
12. **Concept Check** The three-part inequality $a < x < b$ means “ a is less than x and x is less than b .” Which inequality is *not* satisfied by some real number x ?
- A. $-3 < x < 10$ B. $0 < x < 6$
 C. $-3 < x < -1$ D. $-8 < x < -10$

Solve each inequality. Give the solution set in interval notation. See Examples 1 and 2.

13. $-2x + 8 \leq 16$ 14. $-3x - 8 \leq 7$
 15. $-2x - 2 \leq 1 + x$ 16. $-4x + 3 \geq -2 + x$
 17. $3(x + 5) + 1 \geq 5 + 3x$ 18. $6x - (2x + 3) \geq 4x - 5$
 19. $8x - 3x + 2 < 2(x + 7)$ 20. $2 - 4x + 5(x - 1) < -6(x - 2)$
 21. $\frac{4x + 7}{-3} \leq 2x + 5$ 22. $\frac{2x - 5}{-8} \leq 1 - x$
 23. $\frac{1}{3}x + \frac{2}{5}x - \frac{1}{2}(x + 3) \leq \frac{1}{10}$ 24. $-\frac{2}{3}x - \frac{1}{6}x + \frac{2}{3}(x + 1) \leq \frac{4}{3}$

Break-Even Interval Find all intervals where each product will at least break even. See Example 3.

25. The cost to produce x units of picture frames is $C = 50x + 5000$, while the revenue is $R = 60x$.
 26. The cost to produce x units of baseball caps is $C = 100x + 6000$, while the revenue is $R = 500x$.
 27. The cost to produce x units of coffee cups is $C = 105x + 900$, while the revenue is $R = 85x$.
 28. The cost to produce x units of briefcases is $C = 70x + 500$, while the revenue is $R = 60x$.

Solve each inequality. Give the solution set in interval notation. See Example 4.

29. $-5 < 5 + 2x < 11$ 30. $-7 < 2 + 3x < 5$
 31. $10 \leq 2x + 4 \leq 16$ 32. $-6 \leq 6x + 3 \leq 21$
 33. $-11 > -3x + 1 > -17$ 34. $2 > -6x + 3 > -3$
 35. $-4 \leq \frac{x + 1}{2} \leq 5$ 36. $-5 \leq \frac{x - 3}{3} \leq 1$
 37. $-3 \leq \frac{3x - 4}{-5} < 4$ 38. $1 \leq \frac{4x - 5}{-2} < 9$

Solve each quadratic inequality. Give the solution set in interval notation. See Examples 5 and 6.

39. $x^2 - x - 6 > 0$ 40. $x^2 - 7x + 10 > 0$
 41. $2x^2 - 9x \leq 18$ 42. $3x^2 + x \leq 4$
 43. $-x^2 - 4x - 6 \leq -3$ 44. $-x^2 - 6x - 16 > -8$
 45. $x(x - 1) \leq 6$ 46. $x(x + 1) < 12$
 47. $x^2 \leq 9$ 48. $x^2 > 16$

49. $x^2 + 5x + 7 < 0$

50. $4x^2 + 3x + 1 \leq 0$

51. $x^2 - 2x \leq 1$

52. $x^2 + 4x > -1$

53. **Concept Check** Which inequality has solution set $(-\infty, \infty)$?

A. $(x - 3)^2 \geq 0$

B. $(5x - 6)^2 \leq 0$

C. $(6x + 4)^2 > 0$

D. $(8x + 7)^2 < 0$

54. **Concept Check** Which inequality in **Exercise 53** has solution set $\emptyset$?

Solve each rational inequality. Give the solution set in interval notation. See Examples 8 and 9.

55. $\frac{x-3}{x+5} \leq 0$

56. $\frac{x+1}{x-4} > 0$

57. $\frac{1-x}{x+2} < -1$

58. $\frac{6-x}{x+2} > 1$

59. $\frac{3}{x-6} \leq 2$

60. $\frac{3}{x-2} < 1$

61. $\frac{-4}{1-x} < 5$

62. $\frac{-6}{3x-5} \leq 2$

63. $\frac{10}{3+2x} \leq 5$

64. $\frac{1}{x+2} \geq 3$

65. $\frac{7}{x+2} \geq \frac{1}{x+2}$

66. $\frac{5}{x+1} > \frac{12}{x+1}$

67. $\frac{3}{2x-1} > \frac{-4}{x}$

68. $\frac{-5}{3x+2} \geq \frac{5}{x}$

69. $\frac{4}{2-x} \geq \frac{3}{1-x}$

70. $\frac{4}{x+1} < \frac{2}{x+3}$

71. $\frac{x+3}{x-5} \leq 1$

72. $\frac{x+2}{3+2x} \leq 5$

Solve each rational inequality. Give the solution set in interval notation.

73. $\frac{2x-3}{x^2+1} \geq 0$

74. $\frac{9x-8}{4x^2+25} < 0$

75. $\frac{(5-3x)^2}{(2x-5)^3} > 0$

76. $\frac{(5x-3)^3}{(25-8x)^2} \leq 0$

77. $\frac{(2x-3)(3x+8)}{(x-6)^3} \geq 0$

78. $\frac{(9x-11)(2x+7)}{(3x-8)^3} > 0$

(Modeling) Solve each problem.

79. **Box Office Receipts** U.S. movie box office receipts, in billions of dollars, are shown in 5-year increments from 1993 to 2013. (Source: www.boxofficemojo.com)

Year	Receipts
1993	5.154
1998	6.949
2003	9.240
2008	9.631
2013	10.924



1.7 Inequalities

- Linear Inequalities
- Three-Part Inequalities
- Quadratic Inequalities
- Rational Inequalities

An **inequality** says that one expression is greater than, greater than or equal to, less than, or less than or equal to another. As with equations, a value of the variable for which the inequality is true is a solution of the inequality, and the set of all solutions is the solution set of the inequality. Two inequalities with the same solution set are equivalent.

Inequalities are solved with the properties of inequality, which are similar to the properties of equality.

Properties of Inequality

Let a , b , and c represent real numbers.

1. If $a < b$, then $a + c < b + c$.
2. If $a < b$ and if $c > 0$, then $ac < bc$.
3. If $a < b$ and if $c < 0$, then $ac > bc$.

Replacing $<$ with $>$, $\leq$, or $\geq$ results in similar properties. (Restrictions on c remain the same.)

NOTE Multiplication may be replaced by division in Properties 2 and 3. *Always remember to reverse the direction of the inequality symbol when multiplying or dividing by a negative number.*

Linear Inequalities The definition of a *linear inequality* is similar to the definition of a linear equation.

Linear Inequality in One Variable

A **linear inequality in one variable** is an inequality that can be written in the form

$$ax + b > 0, *$$

where a and b are real numbers and $a \neq 0$.

*The symbol $>$ can be replaced with $<$, $\leq$, or $\geq$.

EXAMPLE 1 Solving a Linear Inequality

Solve $-3x + 5 > -7$.

SOLUTION $-3x + 5 > -7$

$$-3x + 5 - 5 > -7 - 5 \quad \text{Subtract 5.}$$

$$-3x > -12 \quad \text{Combine like terms.}$$

Don't forget to reverse the inequality symbol here.

$$\frac{-3x}{-3} < \frac{-12}{-3}$$

$$x < 4$$

Divide by -3 . Reverse the direction of the inequality symbol when multiplying or dividing by a negative number.

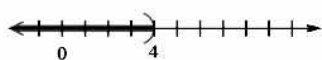


Figure 9

Thus, the original inequality $-3x + 5 > -7$ is satisfied by any real number less than 4. The solution set can be written $\{x | x < 4\}$.

A graph of the solution set is shown in **Figure 9**, where the parenthesis is used to show that 4 itself does not belong to the solution set. As shown below, testing values from the solution set in the original inequality will produce true statements. Testing values outside the solution set produces false statements.

CHECK	$-3x + 5 > -7$	Original inequality
$-3(0) + 5 > -7$	Let $x = 0$.	$-3(5) + 5 > -7$ Let $x = 5$.
$5 > -7$ ✓ True		$-10 > -7$ False

The solution set of the inequality,

$$\{x | x < 4\}, \quad \text{Set-builder notation}$$

is an example of an **interval**. We use **interval notation** to write intervals. With this notation, we write the above interval as

$$(-\infty, 4). \quad \text{Interval notation}$$

The symbol $-\infty$ does not represent an actual number. Rather, it is used to show that the interval includes all real numbers less than 4. The interval $(-\infty, 4)$ is an example of an **open interval** because the endpoint, 4, is not part of the interval. An interval that includes both its endpoints is a **closed interval**. A square bracket indicates that a number *is* part of an interval, and a parenthesis indicates that a number *is not* part of an interval.

✓ **Now Try Exercise 13.**

In the table that follows, we assume that $a < b$.

Summary of Types of Intervals

Type of Interval	Set	Interval Notation	Graph
Open interval	$\{x x > a\}$	(a, ∞)	
	$\{x a < x < b\}$	(a, b)	
	$\{x x < b\}$	$(-\infty, b)$	
Other intervals	$\{x x \geq a\}$	$[a, \infty)$	
	$\{x a < x \leq b\}$	$(a, b]$	
	$\{x a \leq x < b\}$	$[a, b)$	
	$\{x x \leq b\}$	$(-\infty, b]$	
Closed interval	$\{x a \leq x \leq b\}$	$[a, b]$	
Disjoint interval	$\{x x < a \text{ or } x > b\}$	$(-\infty, a) \cup (b, \infty)$	
All real numbers	$\{x x \text{ is a real number}\}$	$(-\infty, \infty)$	

EXAMPLE 2 Solving a Linear InequalitySolve $4 - 3x \leq 7 + 2x$. Give the solution set in interval notation.

SOLUTION

$$\begin{array}{rcl}
 4 - 3x & \leq & 7 + 2x \\
 4 - 3x - 4 & \leq & 7 + 2x - 4 \quad \text{Subtract 4.} \\
 -3x & \leq & 3 + 2x \quad \text{Combine like terms.} \\
 -3x - 2x & \leq & 3 + 2x - 2x \quad \text{Subtract } 2x. \\
 -5x & \leq & 3 \quad \text{Combine like terms.} \\
 \frac{-5x}{-5} & \geq & \frac{3}{-5} \quad \text{Divide by } -5. \text{ Reverse the direction of} \\
 & & \text{the inequality symbol.} \\
 x & \geq & -\frac{3}{5} \quad \frac{a}{-b} = -\frac{a}{b}
 \end{array}$$

**Figure 10**In interval notation, the solution set is $[-\frac{3}{5}, \infty)$. See **Figure 10** for the graph. **Now Try Exercise 15.**

A product will break even, or begin to produce a profit, only if the revenue from selling the product at least equals the cost of producing it. If R represents revenue and C is cost, then the **break-even point** is the point where $R = C$.

EXAMPLE 3 Finding the Break-Even Point

If the revenue and cost of a certain product are given by

$$R = 4x \quad \text{and} \quad C = 2x + 1000,$$

where x is the number of units produced and sold, at what production level does R at least equal C ?

SOLUTION Set $R \geq C$ and solve for x .

$$\begin{array}{rcl}
 R & \geq & C \\
 \text{At least equal} & & \\
 \text{to translates} & & \\
 \text{as } \geq, & & \\
 4x & \geq & 2x + 1000 \quad \text{Substitute.} \\
 2x & \geq & 1000 \quad \text{Subtract } 2x. \\
 x & \geq & 500 \quad \text{Divide by 2.}
 \end{array}$$

The break-even point is at $x = 500$. This product will at least break even if the number of units produced and sold is in the interval $[500, \infty)$.

Now Try Exercise 25.**Three-Part Inequalities**The inequality $-2 < 5 + 3x < 20$ says that

$$5 + 3x \text{ is between } -2 \text{ and } 20.$$

This inequality is solved using an extension of the properties of inequality given earlier, working with all three expressions at the same time.

EXAMPLE 4 Solving a Three-Part InequalitySolve $-2 < 5 + 3x < 20$. Give the solution set in interval notation.

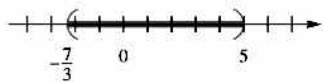
SOLUTION $-2 < 5 + 3x < 20$

$$-2 - 5 < 5 + 3x - 5 < 20 - 5 \quad \text{Subtract 5 from each part.}$$

$$-7 < 3x < 15 \quad \text{Combine like terms in each part.}$$

$$\frac{-7}{3} < \frac{3x}{3} < \frac{15}{3} \quad \text{Divide each part by 3.}$$

$$-\frac{7}{3} < x < 5$$

**Figure 11**The solution set, graphed in **Figure 11**, is the interval $\left(-\frac{7}{3}, 5\right)$. **Now Try Exercise 29.**

Quadratic Inequalities We can distinguish a *quadratic inequality* from a linear inequality by noticing that it is of degree 2.

Quadratic InequalityA **quadratic inequality** is an inequality that can be written in the form

$$ax^2 + bx + c < 0,^*$$

where a , b , and c are real numbers and $a \neq 0$.*The symbol $<$ can be replaced with $>$, $\leq$, or $\geq$.

One method of solving a quadratic inequality involves finding the solutions of the corresponding quadratic equation and then testing values in the intervals on a number line determined by those solutions.

Solving a Quadratic Inequality**Step 1** Solve the corresponding quadratic equation.**Step 2** Identify the intervals determined by the solutions of the equation.**Step 3** Use a test value from each interval to determine which intervals form the solution set.**EXAMPLE 5** Solving a Quadratic InequalitySolve $x^2 - x - 12 < 0$.**SOLUTION****Step 1** Find the values of x that satisfy $x^2 - x - 12 = 0$.

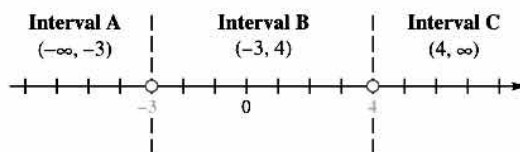
$$x^2 - x - 12 = 0 \quad \text{Corresponding quadratic equation}$$

$$(x + 3)(x - 4) = 0 \quad \text{Factor.}$$

$$x + 3 = 0 \quad \text{or} \quad x - 4 = 0 \quad \text{Zero-factor property}$$

$$x = -3 \quad \text{or} \quad x = 4 \quad \text{Solve each equation.}$$

Step 2 The two numbers -3 and 4 cause the expression $x^2 - x - 12$ to *equal* zero and can be used to divide the number line into three intervals, as shown in **Figure 12**. The expression $x^2 - x - 12$ will take on a value that is either *less than* zero or *greater than* zero on each of these intervals. We are looking for x -values that make the expression *less than* zero, so we use open circles at -3 and 4 to indicate that they are not included in the solution set.



Use open circles because the inequality symbol does not include *equality*. -3 and 4 do not satisfy the *inequality*.

Figure 12

Step 3 Choose a test value in each interval to see whether it satisfies the original inequality, $x^2 - x - 12 < 0$. If the test value makes the statement true, then the entire interval belongs to the solution set.

Interval	Test Value	Is $x^2 - x - 12 < 0$ True or False?
A: $(-\infty, -3)$	-4	$(-4)^2 - (-4) - 12 \stackrel{?}{<} 0$ $8 < 0$ False
B: $(-3, 4)$	0	$0^2 - 0 - 12 \stackrel{?}{<} 0$ $-12 < 0$ True
C: $(4, \infty)$	5	$5^2 - 5 - 12 \stackrel{?}{<} 0$ $8 < 0$ False

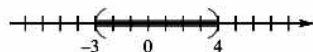


Figure 13

Because the values in Interval B make the inequality true, the solution set is the interval $(-3, 4)$. See **Figure 13**. **Now Try Exercise 41.**

EXAMPLE 6 Solving a Quadratic Inequality

Solve $2x^2 + 5x - 12 \geq 0$.

SOLUTION

Step 1 Find the values of x that satisfy $2x^2 + 5x - 12 = 0$.

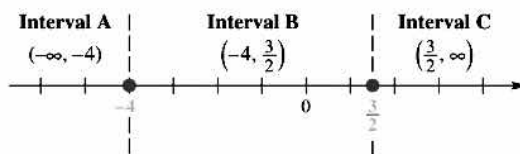
$$2x^2 + 5x - 12 = 0 \quad \text{Corresponding quadratic equation}$$

$$(2x - 3)(x + 4) = 0 \quad \text{Factor.}$$

$$2x - 3 = 0 \quad \text{or} \quad x + 4 = 0 \quad \text{Zero-factor property}$$

$$x = \frac{3}{2} \quad \text{or} \quad x = -4 \quad \text{Solve each equation.}$$

Step 2 The values $\frac{3}{2}$ and -4 cause the expression $2x^2 + 5x - 12$ to equal 0 and can be used to form the intervals $(-\infty, -4)$, $(-4, \frac{3}{2})$, and $(\frac{3}{2}, \infty)$ on the number line, as seen in **Figure 14**.



Use closed circles because the inequality symbol includes *equality*. -4 and $\frac{3}{2}$ satisfy the *inequality*.

Figure 14

Step 3 Choose a test value in each interval.

Interval	Test Value	Is $2x^2 + 5x - 12 \geq 0$ True or False?
A: $(-\infty, -4)$	-5	$2(-5)^2 + 5(-5) - 12 \stackrel{?}{\geq} 0$ $13 \geq 0$ True
B: $(-4, \frac{3}{2})$	0	$2(0)^2 + 5(0) - 12 \stackrel{?}{\geq} 0$ $-12 \geq 0$ False
C: $(\frac{3}{2}, \infty)$	2	$2(2)^2 + 5(2) - 12 \stackrel{?}{\geq} 0$ $6 \geq 0$ True

The values in Intervals A and C make the inequality true, so the solution set is a disjoint interval: the *union* of the two intervals, written

$$(-\infty, -4] \cup \left[\frac{3}{2}, \infty\right).$$

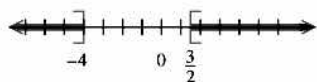


Figure 15

The graph of the solution set is shown in **Figure 15**.

✓ **Now Try Exercise 39.**

NOTE Inequalities that use the symbols $<$ and $>$ are **strict inequalities**, while $\leq$ and $\geq$ are used in **nonstrict inequalities**. The solutions of the equation in **Example 5** were not included in the solution set because the inequality was a *strict* inequality. In **Example 6**, the solutions of the equation were included in the solution set because of the nonstrict inequality.

EXAMPLE 7 Finding Projectile Height

If a projectile is launched from ground level with an initial velocity of 96 ft per sec, its height s in feet t seconds after launching is given by the following equation.

$$s = -16t^2 + 96t$$

When will the projectile be greater than 80 ft above ground level?

SOLUTION

$$-16t^2 + 96t > 80 \quad \text{Set } s \text{ greater than 80.}$$

$$-16t^2 + 96t - 80 > 0 \quad \text{Subtract 80.}$$

$$\text{Reverse the direction of the inequality symbol.} \quad t^2 - 6t + 5 < 0 \quad \text{Divide by } -16.$$

Now solve the corresponding *equation*.

$$t^2 - 6t + 5 = 0$$

$$(t - 1)(t - 5) = 0 \quad \text{Factor.}$$

$$t - 1 = 0 \quad \text{or} \quad t - 5 = 0 \quad \text{Zero-factor property}$$

$$t = 1 \quad \text{or} \quad t = 5 \quad \text{Solve each equation.}$$

Use these values to determine the intervals

$$(-\infty, 1), \quad (1, 5), \quad \text{and} \quad (5, \infty).$$

We are solving a strict inequality, so solutions of the equation $t^2 - 6t + 5 = 0$ are *not* included. Choose a test value from each interval to see whether it satisfies the inequality $t^2 - 6t + 5 < 0$. See **Figure 16** on the next page.

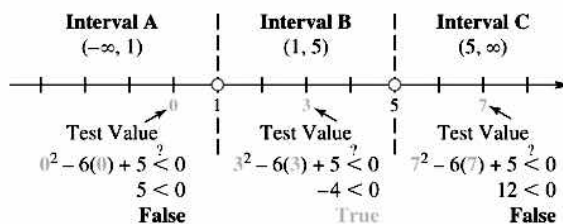


Figure 16

The values in Interval B, $(1, 5)$, make the inequality true. The projectile is greater than 80 ft above ground level between 1 and 5 sec after it is launched.

✓ **Now Try Exercise 81.**

Rational Inequalities Inequalities involving one or more rational expressions are **rational inequalities**.

$$\frac{5}{x+4} \geq 1 \quad \text{and} \quad \frac{2x-1}{3x+4} < 5 \quad \text{Rational inequalities}$$

Solving a Rational Inequality

Step 1 Rewrite the inequality, if necessary, so that 0 is on one side and there is a single fraction on the other side.

Step 2 Determine the values that will cause either the numerator or the denominator of the rational expression to equal 0. These values determine the intervals on the number line to consider.

Step 3 Use a test value from each interval to determine which intervals form the solution set.

A value causing a denominator to equal zero will never be included in the solution set. If the inequality is strict, any value causing the numerator to equal zero will be excluded. If the inequality is nonstrict, any such value will be included.

CAUTION Solving a rational inequality such as $\frac{5}{x+4} \geq 1$ by multiplying each side by $x+4$ requires considering *two cases*, because the sign of $x+4$ depends on the value of x . If $x+4$ is negative, then the inequality symbol must be reversed. The procedure described in the preceding box eliminates the need for considering separate cases.

EXAMPLE 8 Solving a Rational Inequality

Solve $\frac{5}{x+4} \geq 1$.

SOLUTION

Step 1 $\frac{5}{x+4} - 1 \geq 0$ Subtract 1 so that 0 is on one side.

$\frac{5}{x+4} - \frac{x+4}{x+4} \geq 0$ Use $x+4$ as the common denominator.

Note the careful use of parentheses.

$\frac{5 - (x+4)}{x+4} \geq 0$ Write as a single fraction.

$\frac{1-x}{x+4} \geq 0$ Combine like terms in the numerator, being careful with signs.

Step 2 The quotient possibly changes sign only where x -values make the numerator or denominator 0. This occurs at

$$1 - x = 0 \quad \text{or} \quad x + 4 = 0$$

$$x = 1 \quad \text{or} \quad x = -4.$$

These values form the intervals $(-\infty, -4)$, $(-4, 1)$, and $(1, \infty)$ on the number line, as seen in **Figure 17**.

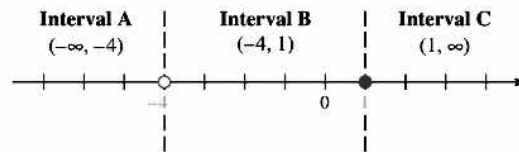


Figure 17

Use a solid circle on 1 because the symbol is $\geq$. The value -4 cannot be in the solution set because it causes the denominator to equal 0. Use an open circle on -4 .

Step 3 Choose test values.

Interval	Test Value	Is $\frac{5}{x+4} \geq 1$ True or False?
A: $(-\infty, -4)$	-5	$\frac{5}{-5+4} \geq 1$ $-5 \geq 1$ False
B: $(-4, 1)$	0	$\frac{5}{0+4} \geq 1$ $\frac{5}{4} \geq 1$ True
C: $(1, \infty)$	2	$\frac{5}{2+4} \geq 1$ $\frac{5}{6} \geq 1$ False

The values in Interval B, $(-4, 1)$, satisfy the original inequality. The value 1 makes the nonstrict inequality true, so it must be included in the solution set. Because -4 makes the denominator 0, it must be excluded. The solution set is the interval $(-4, 1]$. **Now Try Exercise 59.**

CAUTION Be careful with the endpoints of the intervals when solving rational inequalities.

EXAMPLE 9 Solving a Rational Inequality

Solve $\frac{2x-1}{3x+4} < 5$.

SOLUTION $\frac{2x-1}{3x+4} - 5 < 0$ Subtract 5.

$$\frac{2x-1}{3x+4} - \frac{5(3x+4)}{3x+4} < 0 \quad \text{The common denominator is } 3x+4.$$

$$\frac{2x-1-5(3x+4)}{3x+4} < 0 \quad \text{Write as a single fraction.}$$

Be careful with signs. $\frac{2x-1-15x-20}{3x+4} < 0$ Distributive property

$$\frac{-13x-21}{3x+4} < 0 \quad \text{Combine like terms in the numerator.}$$

Set the numerator and denominator of $\frac{-13x - 21}{3x + 4}$ equal to 0 and solve the resulting equations to find the values of x where sign changes may occur.

$$-13x - 21 = 0 \quad \text{or} \quad 3x + 4 = 0$$

$$x = -\frac{21}{13} \quad \text{or} \quad x = -\frac{4}{3}$$

Use these values to form intervals on the number line. Use an open circle at $-\frac{21}{13}$ because of the strict inequality, and use an open circle at $-\frac{4}{3}$ because it causes the denominator to equal 0. See **Figure 18**.

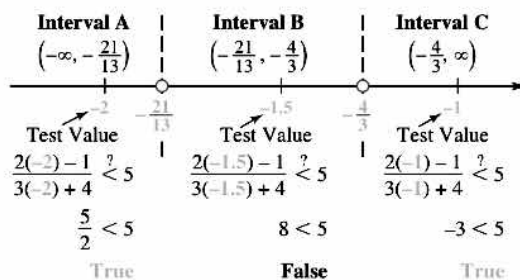


Figure 18

Choosing a test value from each interval shows that the values in Intervals A and C satisfy the original inequality, $\frac{2x-1}{3x+4} < 5$. So the solution set is the union of these intervals.

$$\left(-\infty, -\frac{21}{13}\right) \cup \left(-\frac{4}{3}, \infty\right)$$

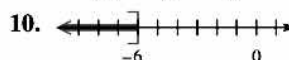
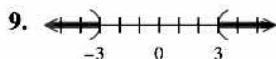
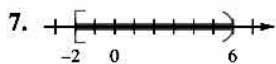
Now Try Exercise 71.

1.7 Exercises

CONCEPT PREVIEW Match the inequality in each exercise in Column I with its equivalent interval notation in Column II.

I

- $x < -6$
- $x \leq 6$
- $-2 < x \leq 6$
- $x^2 \geq 0$
- $x \geq -6$
- $6 \leq x$



II

- $(-2, 6]$
- $[-2, 6)$
- $(-\infty, -6]$
- $[6, \infty)$
- $(-\infty, -3) \cup (3, \infty)$
- $(-\infty, -6)$
- $(0, 8)$
- $(-\infty, \infty)$
- $[-6, \infty)$
- $(-\infty, 6]$